A central limit theorem in the framework of the Thompson group F

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Outline

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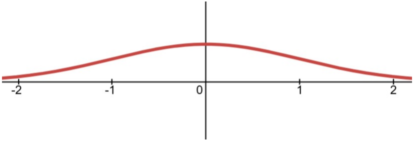
Theorem (Classical Central Limit Theorem)

Suppose (X_n) *is a sequence of IID (independent and identically distributed) random variables with mean* 0 *and variance* 1*. Then*

$$S_n := \frac{X_1 + \cdots + X_n}{\sqrt{n}} \xrightarrow{distr} X,$$

where $\xrightarrow{distr}$ *denotes convergence in distribution and* X *is a random variable with the standard normal distribution* $\mathcal{N}(0,1)$*.*



$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

Definition (Convergence in distribution)

The sequence of random variables (Y_n) is said to converge in distribution to a random variable Y, denoted by

$$Y_n \xrightarrow{\text{distr}} Y$$

if the respective cumulative distribution functions converge, that is,

$$F_{Y_n}(x) \to F_Y(x), \text{ wherever } F_Y \text{ is continuous.}$$

This is equivalent to the convergence of the expected value operators:

$$\mathbb{E}(f(Y_n)) \to \mathbb{E}(f(Y)), \quad \text{for all bounded continuous functions } f.$$

A (noncommutative) $*$-**probability space** is a pair $(\mathcal{M}, \psi)$, where $\mathcal{M}$ is a unital $*$-algebra and ψ is a unital positive linear functional on $\mathcal{M}$.

Unital $*$-algebra: An algebra $\mathcal{M}$ with an identity element which has an involution $*$.
Unital positive linear functional ψ: $\psi(1) = 1$, $\psi(x^*x) \geq 0$ for all $x \in \mathcal{M}$, $\psi(\alpha x + y) = \alpha\psi(x) + \psi(y)$, for all $x, y \in \mathcal{M}$ and $\alpha \in \mathbb{C}$.

Examples

- $(M_n(\mathbb{C}), \mathrm{Tr})$, where Tr is the normalized trace on square matrices.
- $(\mathbb{C}G, \varphi)$ where $\mathbb{C}G$ is the group algebra of a group G and $\varphi = \delta_{g=e}$ is the canonical trace.

How does a classical probability space fit in?

Let (Ω, Σ, μ) be a probability space, where Ω is a sample space, Σ is a σ-algebra and μ is a probability measure.
Then $\mathcal{L} := L^\infty(\Omega, \Sigma, \mu)$ is a commutative $*$-algebra and

$$\mathrm{tr}_\mu(f) := \int_\Omega f \, d\mu$$

defines a unital positive linear functional on $\mathcal{L}$. The pair $(\mathcal{L}, \mathrm{tr}_\mu)$ is a $*$-probability space.
Sometimes we need to work with the $*$-algebra

$$L^{\infty-}(\Omega, \Sigma, \mu) := \bigcap_{p \geq 1} L^p(\Omega, \Sigma, \mu)$$

In a $*$-probability space $(\mathcal{M}, \psi)$, a (noncommutative) random variable is an element x in $\mathcal{M}$.

Definition (Convergence in distribution)

Let $(\mathcal{A}_n, \varphi_n)$ $(n \in \mathbb{N})$ and $(\mathcal{A}, \varphi)$ be $*$-probability spaces. Let $a_n = a_n^* \in \mathcal{A}_n$ $(n \in \mathbb{N})$ and $x \in \mathcal{A}$ be random variables. We say that (a_n) converges in distribution to x as $n \to \infty$, and denote this by $a_n \xrightarrow{\text{distr}} x$, if we have

$$\lim_{n\to\infty} \varphi_n(a_n^d) = \varphi(x^d), \ \forall d \in \mathbb{N}.$$

Definition (Tensor independence)

A sequence of random variables $(a_n) \subset \mathcal{M}$ is said to be tensor independent if $a_i a_j = a_j a_i$ for all $i, j \in \mathbb{N}$ and

$$\varphi(a_{n_1}^{m_1} \cdots a_{n_k}^{m_k}) = \varphi(a_{n_1}^{m_1}) \cdots \varphi(a_{n_k}^{m_k}), \ n_l, m_l \in \mathbb{N}_0, k \in \mathbb{N}.$$

Theorem (Classical CLT restated algebraically)

Let $(\mathcal{A}, \varphi)$ be a $*$-probability space where $\mathcal{A}$ is commutative. Suppose $(a_n = a_n^*) \subset \mathcal{A}$ is a sequence of tensor independent and identically distributed random variables. Furthermore, assume that $\varphi(a_n) = 0$ and $\varphi(a_n^2) = 1$ $(n \in \mathbb{N})$. Then we have

$$s_n := \frac{a_1 + \cdots + a_n}{\sqrt{n}} \xrightarrow{distr} x,$$

where x is a normally distributed random variable of mean 0 and variance 1.

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Remark: As the normal distribution is determined by its moments, this seemingly weaker algebraic formulation is actually equivalent to the classical CLT stated earlier.

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Definition (Free independence)

A sequence of random variables $(a_n) \subset \mathcal{A}$ is said to be freely independent if for $\mathcal{A}_i := * - \text{alg}\{1, a_i\}$, we have $\varphi(x_1 \cdots x_k) = 0$ for $k \in \mathbb{N}$ whenever

1. $\varphi(x_j) = 0$ for all $j \in \{1, \dots, k\}$;
2. $x_j \in \mathcal{A}_{i(j)}$ for each $j \in \{1, \dots, k\}$;
3. $i(1) \neq i(2), i(2) \neq i(3), \dots, i(k-1) \neq i(k)$.

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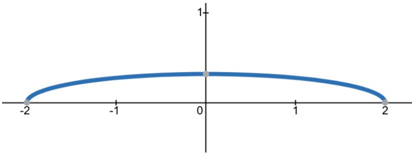
Remark: The joint distributions of a freely independent sequence of random variables is determined by the knowledge of the individual distributions of each random variable.

Theorem (Free CLT, Voiculescu 1985, Combinatorial proof by Speicher 1990)

Let $(\mathcal{A}, \varphi)$ *be a* $*$*-probability space and* $(a_n = a_n^*) \subset \mathcal{A}$ *be a sequence of* *freely* *independent and identically distributed random variables. Furthermore, assume that* $\varphi(a_n) = 0$ *and* $\varphi(a_n^2) = 1$ $(n \in \mathbb{N})$. *Then we have*

$$s_n := \frac{a_1 + \cdots + a_n}{\sqrt{n}} \xrightarrow{distr} s,$$

where s *is a semicircular element of mean* 0 *and variance* 1.



$$f(x) = \frac{1}{2\pi}\sqrt{4 - x^2}$$

In both central limit theorems, we try to find

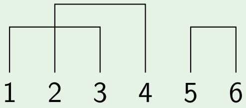
$$\lim_{n\to\infty} \varphi(s_n^d) = \lim_{n\to\infty} \frac{1}{n^{d/2}} \sum_{\underline{i}:[d]\to\{1,\dots,n\},} \varphi\Big(a_{\underline{i}(1)} \cdots a_{\underline{i}(d)}\Big)$$

- Both tensor independence and free independence give a rule for calculating mixed moments from the values of the moments. This means $\varphi(a_{\underline{i}(1)} \cdots a_{\underline{i}(d)})$ depends on the tuple $(\underline{i}(1), \dots, \underline{i}(d))$ only through the information on which indices are the same and which are different.
- This information is encoded via a partition – the kernel of the tuple $\underline{i}$. Here, $\ker(\underline{i})$ is the partition of $[d]$ into the level sets of the tuple $\underline{i}$. $\varphi(a_3 a_1 a_3 a_2 a_2 a_4 a_1 a_4) \rightsquigarrow \pi = \ker(\underline{i}) = \{\{1,3\},\{2,7\},\{4,5\},\{6,8\}\}$.
- For both tensor and free independence, the only tuples that contribute are those whose kernels are pair partitions – *all* pair partitions in the tensor case, and *non-crossing* pair partitions in the free case.

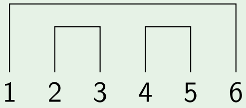
Pair partitions: For even $d \in \mathbb{N}$, a pair partition of the set $[d] := \{1, \ldots, d\}$ is a disjoint collection of pairs, i.e., $\pi = \{V_1, \ldots, V_{\frac{d}{2}}\}$ with $V_i \subseteq [d]$, $V_i \cap V_j = \varnothing$ for $i \neq j$, and $|V_i| = 2$ for each $i \in [\frac{d}{2}]$. The set of pair partitions of $[d]$ is denoted by $\mathcal{P}_2(d)$.

Example

A crossing pair partition of $[6]$: $\{\{1,3\},\{2,4\},\{5,6\}\}$



A non-crossing pair partition of $[6]$: $\{\{1,6\},\{2,3\},\{4,5\}\}$



Let d be even.

- The number of all pair partitions of $[d]$ is

$$(d-1)!! := (d-1) \times (d-3) \times \cdots (3) \times (1).$$

- The number of non-crossing pair partitions of $[d]$ is given by the Catalan number

$$C_{\frac{d}{2}} = \frac{1}{(\frac{d}{2}+1)} \binom{d}{\frac{d}{2}}.$$

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$$C_{\frac{d}{2}} = \frac{1}{(\frac{d}{2}+1)} \binom{d}{\frac{d}{2}}.$$

  This is the d-th moment of Wigner's semicircular distribution, which is the limiting law in the free CLT.

Theorem (Generalized CLT, Bożejko, Speicher, von Waldenfels 1994)

Let $(\mathcal{A}, \varphi)$ *be a* $*$*-probability space and suppose that the sequence* $(a_n = a_n^*)_{n=1}^{\infty} \subset \mathcal{A}$ *satisfies:*

1. *an invariance principle known as exchangeability.*
2. *the singleton vanishing property.*

Let $s_n = \frac{1}{\sqrt{n}}(a_1 + \cdots + a_n)$. *Then one has, for any* $d \in \mathbb{N}$,

$$\lim_{n \to \infty} \varphi(s_n^d) = \sum_{\pi \in \mathcal{P}_2(d)} \varphi_\pi$$

where $\varphi_\pi = \varphi(a_{\underline{\ell}(1)} \cdots a_{\underline{\ell}(d)})$ *with* $\pi = \ker(\underline{\ell})$.

Outline

Can we compute the limit distribution of a sequence in the group algebra of a particular group of interest?

- Infinite symmetric group S_∞ (Biane 1995; Campbell, Köstler, Nica 2021 and 2022).
- Coxeter and Artin systems of extra large types (Fendler 2003).

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Thompson's group F has the following infinite presentation:

$$F = \langle g_0, g_1, \dots \mid g_n g_k = g_k g_{n+1},\ 0 \le k < n < \infty \rangle.$$

For instance, $g_7 g_3 = g_3 g_8$.
Let $(\mathbb{C}(F), \varphi)$ be the $*$-probability space with $\mathbb{C}(F)$ denoting the group algebra of F and φ the canonical trace defined by

$$\varphi(x) = \begin{cases} 1, & x = e \\ 0, & x \neq e. \end{cases}$$

- The group F was introduced by Richard Thompson in 1965 as a certain subgroup of piece-wise linear homeomorphisms on the interval $[0, 1]$.
- It can also be realised in terms of morphisms on the category of rooted ordered binary trees.
- It is an object of interest to those in many fields; notably, the question of its amenability is still open.
- There has also been interest in probabilistic aspects of F.

Theorem (A Central Limit Theorem for F, K. 2023)

Let (a_n) be the sequence of self-adjoint random variables in $(\mathbb{C}(F), \varphi)$ given by

$$a_n = \frac{g_n + g_n^*}{\sqrt{2}}, \;\; n \in \mathbb{N}_0$$

and

$$s_n := \frac{1}{\sqrt{n}}(a_0 + \cdots + a_{n-1}), \;\; n \in \mathbb{N}.$$

Then we have

$$\lim_{n\to\infty} \varphi(s_n^d) = \begin{cases} (d-1)!! & \text{for } d \text{ even,} \\ 0 & \text{for } d \text{ odd.} \end{cases}$$

That is,

$$s_n \xrightarrow{distr} x,$$

where x is a random variable with standard normal distribution.

The d-th moment of s_n can be expressed as

$$\varphi(s_n^d) = \frac{1}{(2n)^{d/2}} \sum_{\substack{\underline{i}:[d]\to\{0,\ldots,n-1\},\\ \underline{\varepsilon}:[d]\to\{-1,1\}}} \varphi\left(g_{\underline{i}(1)}^{\underline{\varepsilon}(1)} \cdots g_{\underline{i}(d)}^{\underline{\varepsilon}(d)}\right)$$
$$= \frac{1}{(2n)^{d/2}} |\mathcal{W}_0(d,n)|$$

where

$$\mathcal{W}_0(d,n) := \left\{ (\underline{i}, \underline{\varepsilon}) \;\middle|\; \begin{array}{l} \underline{i} : [d] \to \{0,\ldots,n-1\},\ \underline{\varepsilon} : [d] \to \{-1,1\}, \\ \text{and } g_{\underline{i}(1)}^{\underline{\varepsilon}(1)} \cdots g_{\underline{i}(d)}^{\underline{\varepsilon}(d)} = e \end{array} \right\}.$$

Task: Count the number of words (of length d) composed of the first n generators of F and their inverses which evaluate to the identity.

Outline

Step 0: We observe that

- Only words of *even* length d can evaluate to the identity. So all odd moments are 0.
- Each word which evaluates to the identity must have an equal number of generators and inverses.

Step 1: We reduce each such word $(\underline{i}, \underline{\varepsilon})$ to a normal form

$$(\underline{j}, \underline{\varepsilon}_0) = (g_{\underline{j}(1)}, \dots g_{\underline{j}(\frac{d}{2})}, g^{-1}_{\underline{j}(\frac{d}{2})}, \dots, g^{-1}_{\underline{j}(1)})$$

with $\underline{j}(l) + 1 \geq \underline{j}(l+1)$ for all $l \in [\frac{d}{2}]$.

Example

Suppose $d = 8$ and $(\underline{i}, \underline{\varepsilon}) \in \mathcal{W}_0(8, 20)$ is given by

$$(\underline{i}, \underline{\varepsilon}) = (g_2, g_0, g_{18}^{-1}, g_4^{-1}, g_0^{-1}, g_{16}, g_3, g_2^{-1}).$$

Then the normal form $(\underline{j}, \underline{\varepsilon}_0)$ of $(\underline{i}, \underline{\varepsilon})$ is

$$(\underline{j}, \underline{\varepsilon}_0) = (g_{16}, g_2, g_3, g_0, g_0^{-1}, g_3^{-1}, g_2^{-1}, g_{16}^{-1}).$$

The uniqueness of the above normal form can be shown using the formalism of abstract reduction systems.

Definition

An abstract reduction system is a pair $(A, \to)$, where A is a set and $\to$ is a relation on A.

- The element y is a normal form of x if $x \xrightarrow{*} y$ and y cannot be reduced further.
- A reduction is said to be *normalizing* if every element has a normal form.
- A reduction is said to be *terminating* if there is no infinite chain $x_0 \to x_1 \to x_2 \to \cdots$.
- A reduction $\to$ is said to be *confluent* if for all $w, x, y \in A$ with $w \xrightarrow{*} x$ and $w \xrightarrow{*} y$, we have $x \downarrow y$. It is said to be *locally confluent* if for all $w, x, y \in A$ with $w \to x$ and $w \to y$, we have $x \downarrow y$.

Lemma (Newman)

A terminating reduction is confluent if and only if it is locally confluent.

Example

Let $A := \mathbb{N} \setminus \{1\}$ and $\rightarrow := \{(m, n) \mid m > n \text{ and } n|m\}$. Then we have the following:

- m is in normal form if and only if m is prime.
- p is a normal form of m if and only if p is a prime factor of m.
- $\rightarrow$ is terminating as $m \rightarrow n$ implies that $n < m$.
- $\rightarrow$ is normalizing but normal forms are not unique. For example, 2 and 3 are both normal forms of 6.
- $\rightarrow$ is not locally confluent (and hence not confluent). For example, $6 \rightarrow 2$ and $6 \rightarrow 3$ but $2 \not\downarrow 3$.

- We will take A to be the set of words in F and $\to$ to be a reduction described by the relations satisfied by the generators of F.
- We define two reduction rules in turn:

$$(g_k^{-1}, g_l) \to \begin{cases} (g_l, g_k^{-1}) & \text{if } k = l, \\ (g_l, g_{k+1}^{-1}) & \text{if } k > l, \\ (g_{l+1}, g_k^{-1}) & \text{if } k < l. \end{cases}$$

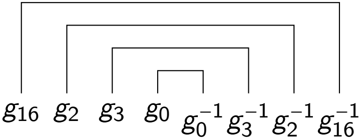
followed by

$$g_k g_l \to g_{k-1} g_l \qquad \text{if } k - 1 > l,$$
$$g_l^{-1} g_k^{-1} \to g_k^{-1} g_{l-1}^{-1} \qquad \text{if } l - 1 > k.$$

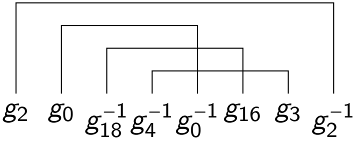
- By showing that the reduction is *terminating* and *(locally) confluent*, we can conclude that each word has a unique normal form.

Step 2: We use the normal form to assign a unique pair partition $\pi \in \mathcal{P}_2(d)$ to each word that evaluates to the identity.

We visualize the rainbow pair partition on the normal form as



We can then trace back to the original word $(\underline{i}, \underline{\varepsilon})$ to get the pair partition $\{\{1,8\},\{2,5\},\{3,6\},\{4,7\}\}$:



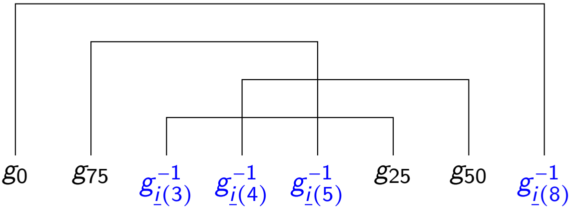
Step 3: Let $N(d,n,\pi)$ be the number of words $(\underline{i},\underline{\varepsilon})$ which are assigned to the pair partition $\pi \in \mathcal{P}_2(d)$. We will find upper and lower bounds on this number.

We will then be able to estimate the limit as $n \to \infty$ of the d-th moment:

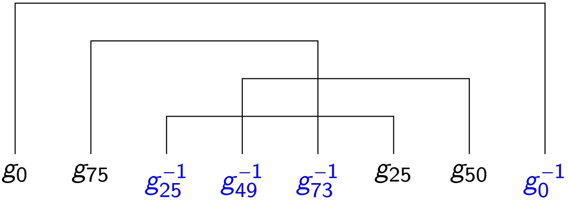
$$\varphi(s_n^d) = \frac{1}{(2n)^{d/2}} |\mathcal{W}_0(d,n)| = \frac{1}{(2n)^{d/2}} \sum_{\pi \in \mathcal{P}_2(d)} N(d,n,\pi).$$

Remark: The second of the above equations is possible thanks to the normal form given by the abstract reduction system.

Key observation: If we know all the generators in a word assigned to some pair partition, and their indices are "sufficiently spread out", then inverse generators can be filled in uniquely to give a word that evaluates to the identity.



Our filled in word is



Recall that the generator with smaller index remains unchanged, for instance,

$$g_0 g_{75} = g_{74} g_0.$$

The precise change in an index is determined by the number of crossings between pairs.

Upshot: We have the freedom to choose only $\frac{d}{2}$ indices from the pool of n indices $\{0, \ldots, n-1\}$ for a word of length d to evaluate to the identity.

Proposition

We get the following bounds on $N(d,n,\pi)$:

$$2^{\frac{d}{2}}\left(\frac{d}{2}\right)!\binom{n+2d-\frac{3d^2}{2}}{\frac{d}{2}} \leq N(d,n,\pi) \leq 2^{\frac{d}{2}}\left(\frac{d}{2}\right)!\binom{n+\frac{d^2}{2}-2}{\frac{d}{2}}. \qquad (1)$$

Recall that we have the limit of the d-th moment of s_n given by

$$\varphi(s_n^d) = \frac{1}{(2n)^{d/2}} \sum_{\pi \in \mathcal{P}_2(d)} N(d,n,\pi).$$

Final Steps:

- Divide (1) by $(2n)^{\frac{d}{2}}$ to get

$$\frac{1}{n^{\frac{d}{2}}} \frac{(n+2d-\frac{3d^2}{2})!}{(n+2d-\frac{3d^2}{2}-\frac{d}{2})!} \le \frac{1}{(2n)^{\frac{d}{2}}} N(d,n,\pi) \le \frac{1}{n^{\frac{d}{2}}} \frac{(n+\frac{d^2}{2}-2)!}{(n+\frac{d^2}{2}-2-\frac{d}{2})!}.$$

- Summing over all pair partitions $\pi \in \mathcal{P}_2(d)$ and in the limit as $n \to \infty$:

$$\lim_{n\to\infty} \frac{1}{n^{\frac{d}{2}}} \sum_{\pi\in\mathcal{P}_2(d)} \frac{(n+2d-\frac{3d^2}{2})!}{(n+2d-\frac{3d^2}{2}-\frac{d}{2})!} \le \lim_{n\to\infty} \sum_{\pi\in\mathcal{P}_2(d)} \frac{1}{(2n)^{\frac{d}{2}}} N(d,n,\pi)$$
$$= \lim_{n\to\infty} \varphi(s_n^d)$$
$$\le \lim_{n\to\infty} \frac{1}{n^{\frac{d}{2}}} \sum_{\pi\in\mathcal{P}_2(d)} \frac{(n+\frac{d^2}{2}-2)!}{(n+\frac{d^2}{2}-2-\frac{d}{2})!}.$$

Final Steps: This gives

$$\sum_{\pi \in \mathcal{P}_2(d)} 1 \leq \lim_{n\to\infty} \varphi(s_n^d) \leq \sum_{\pi \in \mathcal{P}_2(d)} 1.$$

Hence

$$\lim_{n\to\infty} \varphi(s_n^d) = |\mathcal{P}_2(d)| = (d-1)!!.$$

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$$s_n := \frac{1}{\sqrt{n}}(a_0 + \cdots + a_{n-1}), \;\; n \in \mathbb{N}.$$

Then $s_n \xrightarrow{\text{distr}} x$, where x is a normally distributed random variable of mean 0 and variance 1.

Ongoing Research / Some Questions

- What property of the relations satisfied by the generators of F allow us to make this association with pair partitions? Note that the sequence a_n is not exchangeable and does not satisfy the singleton vanishing property (conditions in the Generalized CLT).
- Recently Aiello extended this CLT to Brown - Thompson groups ($g_n g_k = g_k g_{n+p-1}$, $p \geq 2$).
- Can we formulate a multidimensional version of the theorem in the hope of getting the multivariate normal distribution in the limit?

Thank You!