Some trigonometric identities associated with the roots of unity

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This is a corrected and extended version of the slides used in the presentation; minor corrections made on Feb. 17, 2022. More importantly, an observation by Doron Zeilberger, communicated after the presentation, completed the proof of the "vanishing-identity conjecture" on Feb. 17, 2022; two pertinent extra slides have been added after the bibliography.

Optimal Social Distancing on $\mathbb{S}^{\mathrm{d}}$ for $\mathrm{d} \in \mathbb{N}$

- For $N \geq 2$, let
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- Optimal arithmetic and geometric mean distances:

$$\mathcal{A}(N) := \max_{\omega^{(N)}} AMD(\omega^{(N)}) \quad ; \quad \mathcal{G}(N) := \max_{\omega^{(N)}} GMD(\omega^{(N)})$$

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4. These and other optimal configurations ($\longrightarrow$ "optimal" for other purposes) are all associated with minimization of some arithmetic mean pair energy over configurations $\omega^{(N)}$!

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- $1 - AMD(\omega^{(N)})$ and $-\ln GMD\left(\omega^{(N)}\right)$ are, respectively, the special cases $s = -1$ and $s = 0$ of the average s-Riesz pair energy of $\omega^{(N)}$, defined as the arithmetic mean

$$\langle V_s \rangle \left(\omega^{(N)}\right) := \frac{2}{N(N-1)} \sum\sum_{1 \le i < j \le N} V_s\left(D(\mathbf{p}_i, \mathbf{p}_j)\right)$$

of the s-Riesz pair energies $V_s(\mathbf{p}_i, \mathbf{p}_j)$, defined next.

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   - Given $r > 0$, we have $s < t \implies V_s(r) \leq V_t(r)$, with "$=$" iff $r = 1$.

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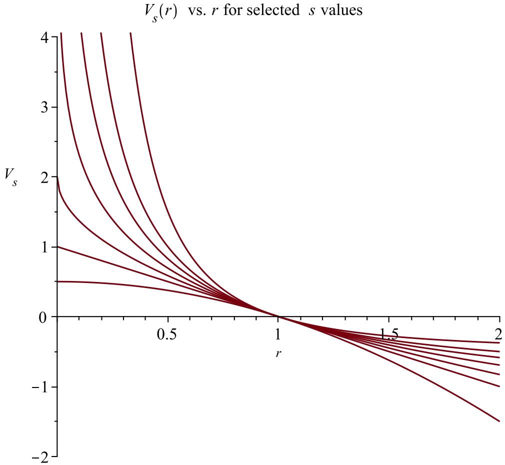

Shown is the Riesz pair energy as function of pair separation for the Riesz parameter values $s \in \{-2, -1, -0.5, 0, 0.5, 1, 2\}$

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- When $s < \mathrm{d}$, the map $N \mapsto v_s(N)$ is bounded above.
- The map $N \mapsto v_{-2}(N)$ is explicitly known for all $\mathrm{d} \in \mathbb{N}$:

$$v_{-2}(N) = -\frac{1}{2}\,\frac{N+1}{N-1} \qquad \text{for} \quad N \geq 2.$$

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- Minimizing N-point configurations ω_N^s exist for $s \geq -2$.

Some optimizing configurations ω_N^s for $\mathrm{d} \in \{1, 2\}$

For $\mathrm{d} = 1$, $s > -2$, and $N \geq 2$, minimizer is rigorously known:

ω_N^s : two anti-podal points ($N = 2$) & regular N-gon ($N \geq 3$).

For $\mathrm{d} = 2$ the following is partly rigorous, partly empirical:

ω_2^s	:	two antipodal points	$s \in (-2, \infty)$;
ω_3^s	:	equatorial equilateral triangle	$s \in (-2, \infty)$;
ω_4^s	:	regular tetrahedron	$s \in (-2, \infty)$;
ω_5^s	:	triangular bi-pyramid	$s \in (-2, 15.04807...]$,
		square pyramid ($f = 1$)	$s \in [15.04807..., \infty)$;
ω_6^s	:	regular octahedron	$s \in (-2, \infty)$;
ω_7^s	:	$C_2(1^1 2^3)(f = 5)$	$s \in (-2, 0]$,
		pentagonal bi-pyramid	$s \in [0, 2]$,
		$C_2(1^1 2^3)(f = 5)$	$s \in [2, 5]$,
		$C_{2v}(1^1 4^1 2^1)(f = 3)$	$s \in [5, 5.5979...]$,
		$C_{3v}(1^1 3^2)(f = 2)$	$s \in [5.5979..., \infty)$.

Empirical Optimal Average s-Riesz Pair Energies, $d = 2$

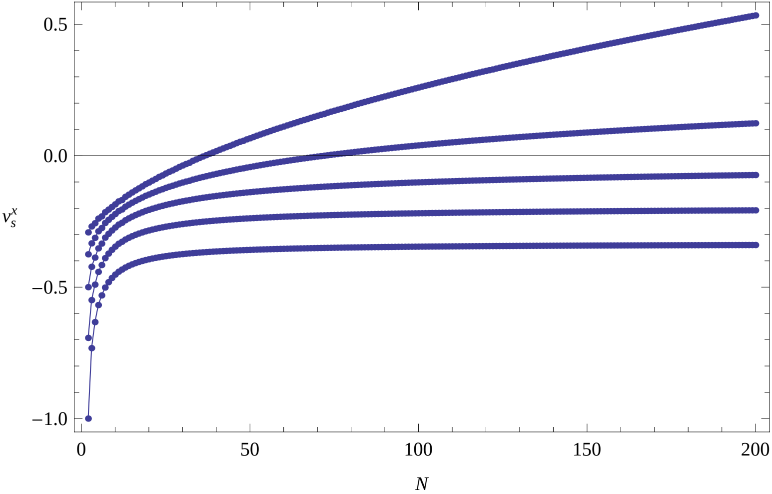

Figure: The empirical functions $v_s^x(N)$ vs. N for $s \in \{-1, 0, 1, 2, 3\}$.

Is $N \mapsto v_s(N)$ concave for each $N \geq 3$ and all $s \geq -2$?

Asymptotic large-N analysis by Brauchart, Hardin, and Saff reveals: The map $N \mapsto v_s(N)$ defined with $\omega_N^s \subset \mathbb{S}^{\mathrm{d}}$ is

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So $N \mapsto v_s(N)$ can only be concave for each $N \geq 3$ if $s \leq 2\mathrm{d}$.

Thus the question now becomes:

Is $N \mapsto v_s(N)$ concave at each $N \geq 3$ when $-2 \leq s \leq 2\mathrm{d}$?

Is $N \mapsto v_s(N)$ concave for each $N \geq 3$ and $-2 \leq s \leq 2d$?

1. Treating N as continuous, the classical second derivative of $N \mapsto v_{-2}(N)$ is

$$\frac{d^2}{dN^2}\Big(-\frac{1}{2}\frac{N+1}{N-1}\Big) = -\frac{2}{(N-1)^3} < 0,$$

so $N \mapsto v_{-2}(N)$ is strictly concave. True for all $\mathrm{d} \in \mathbb{N}$.

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$$\ddot{v}_s^x(N) \equiv v_s^x(N+1) - 2v_s^x(N) + v_s^x(N-1)$$

Nerrattini, Brauchart, M.K. inspected it for $N \in \{3, ..., 199\}$ and $s \in \{-1, 0, 1, 2, 3\}$, and $\mathrm{d} = 2$. **Empirically we found**:

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- The map $N \mapsto v_{-1}^x(N)$ is strictly concave on $3 \leq N \leq 199$;
- For $s \in \{0, 1, 2, 3\}$ there are convexity defects;
- The set of convexity defects grows with s.

For which $s > -2$ is $N \mapsto v_s(N)$ concave for $N \geq 3$?

NBK proved the occurrence of convexity defects in $N \mapsto v_s(N)$ when $\mathrm{d} = 2$ for $s \in \{0, 1, 2, 3\}$, but could not prove the strict concavity of $N \mapsto v_{-1}(N)$ when $N \geq 3$ and $\mathrm{d} = 2$.
On the other hand, $N \mapsto v_{-2}(N)$ is strictly locally concave for $N \geq 3$ and $\mathrm{d} = 2$. (This explicitly known function of N is strictly concave for $N > 1$ with N treated as continuous variable.)
Since $s \mapsto v_s(N)$ is continuous for all $N \geq 2$, the second discrete derivative of $N \mapsto v_s(N)$ is continuous in s too. Hence, in some right neighborhood of $s = -2$ the map $N \mapsto v_s(N)$ must be strictly locally concave.

Open question: How large is this s-neighborhood of $s = -2$?

Since the answer seems hard to find for $\mathrm{d} = 2$, let's look at $\mathrm{d} = 1$!

The average s-Riesz pair energy of the roots of unity

By some general results of Cohn and Kumar (2014), all $d = 1$ N-point optimizers are known for $s > -2$ and can be identified with the N-th roots of unity, after at most $O(2)$ and S_N actions.

The average s-Riesz pair energy of the N-th roots of unity, $\omega_1^{(N)}$, with $N \geq 2$, is easily computed to be given by

$$\boxed{\langle V_s\rangle\big(\omega_1^{(N)}\big) = \frac{1}{s}\left(-1 + \frac{1}{N-1}\sum_{k=1}^{N-1}\frac{1}{2^s\sin^s\big(\frac{k\pi}{N}\big)}\right)}.$$

The r.h.s. is well-defined for $s \in \mathbb{R}$ (with $s = 0$ understood as limit $s \to 0$), yet the N-th roots of unity are not optimal N-point configurations if $s < -2$ and N is even. When N is odd, there may be a left neighborhood of $s = -2$ for which the N-th roots of unity **are** the optimal N-point configuration. (The 3rd roots of unity are the optimal 3 point configuration for $s > \frac{\ln(4/9)}{\ln(4/3)}$.)

Some trigonometric sums from Gradshteyn-Ryzhik

In "Tables of Integrals, Series, and Products" by Gradshteyn and Ryzhik one finds some functions of x expressed as sums of powers of sine over the set of angles that we are concerned with. By evaluating these at $x \to 0$ we find

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- $s = -1$ corresponds to:

$$\sum_{k=1}^{N-1} \sin\left(\frac{k\pi}{N}\right) = \cot\left(\frac{\pi}{2N}\right) \qquad (1.344.1)$$

Concavity of $N \mapsto v_s(N)$, $N \geq 3$, $s \in \{-2, -1\}$, $\mathrm{d} = 1$

- The sum of sine-squares for $s = -2$ reproduces what we knew already:

$$v_{-2}(N) = -\frac{1}{2}\frac{N+1}{N-1}.$$

As found earlier using calculus, $N \mapsto v_{-2}(N)$ is concave.

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$$v_{-2}(N) = -\frac{1}{2}\frac{N+1}{N-1}.$$
As found earlier using calculus, $N \mapsto v_{-2}(N)$ is concave.
- The sum of sines for $s = -1$ allows us to prove the concavity of the map $N \mapsto v_{-1}(N)$ when $\mathrm{d} = 1$, which evaluates to
$$v_{-1}(N) = 1 - \frac{2}{N-1}\cot\left(\frac{\pi}{2N}\right).$$
The proof of concavity is accomplished by treating N as continuous variable and using calculus.

A trigonometric product from Gradshteyn-Ryzhik

In "Tables of Integrals, Series, and Products" by Gradshteyn and Ryzhik one finds some functions of x expressed as products of powers of sine over the set of angles that we are concerned with. By evaluating one of these at $x \to 0$ we find

A trigonometric product from Gradshteyn-Ryzhik

In “Tables of Integrals, Series, and Products” by Gradshteyn and Ryzhik one finds some functions of x expressed as products of powers of sine over the set of angles that we are concerned with. By evaluating one of these at $x \to 0$ we find

- $s = 0$ corresponds to:

$$\prod_{k=1}^{N-1} \sin\left(\frac{k\pi}{N}\right) = \frac{N}{2^{N-1}} \tag{1.392.1}$$

Concavity of $N \mapsto v_s(N)$, $N \geq 3$, $s = 0$, $\mathrm{d} = 1$

The product of sines for $s = 0$ allows us to prove the concavity of the map $N \mapsto v_0(N)$ when $\mathrm{d} = 1$, which evaluates to

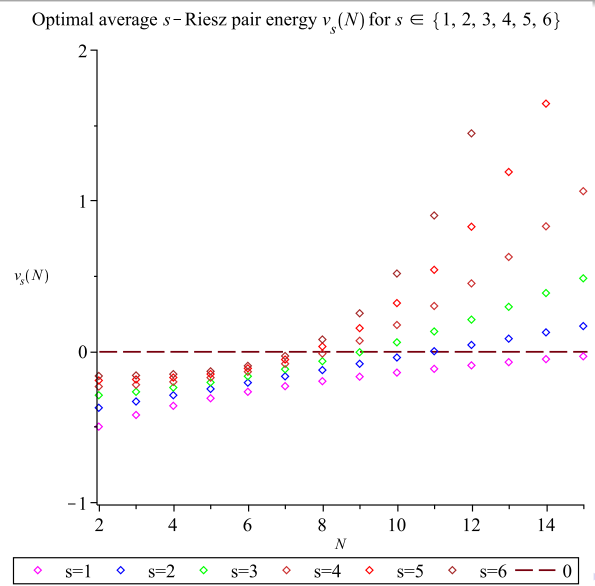
$$v_0(N) = -\frac{\ln N}{N-1}.$$

This is easily proven to be concave by treating N as continuous variable and using calculus.

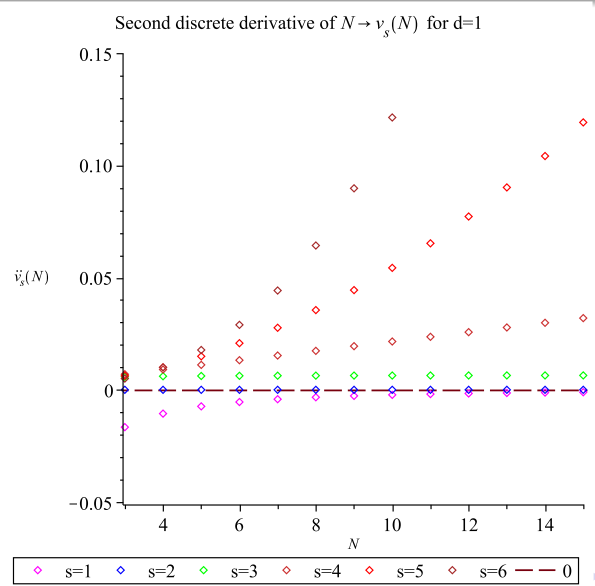
MAPLE comes to aid for when $s > 0$

No formulas were found in Gradshteyn-Ryzhik that would correspond to $s > 0$. Using MAPLE we plotted $N \mapsto v_s(N)$ and $N \mapsto \ddot{v}_s(N)$, and some higher discrete derivatives, for a selection of s values, as described in the following.

MAPLE-generated plots of $N \mapsto v_s(N)$ for $s > 0$



MAPLE-generated plots of $N \mapsto \ddot{v}_s(N)$ for $s > 0$

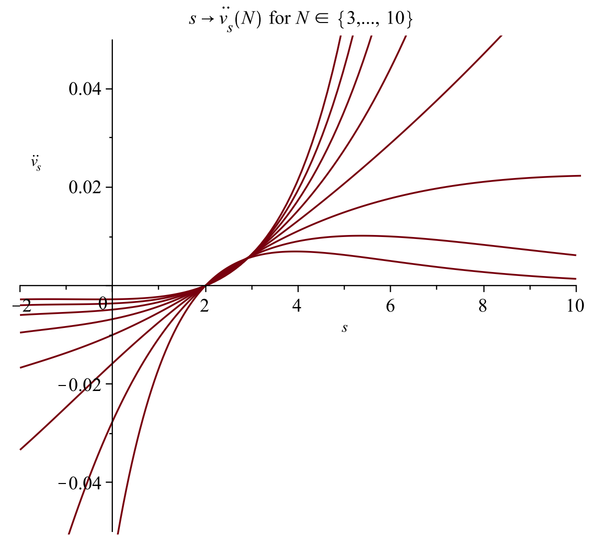


A MAPLE plots-inspired Conjecture about average s-Riesz pair energies of the roots of unity

Conjecture 1: The average s-Riesz pair energy of the roots of unity, i.e. $N \mapsto v_s(N)$ when $\mathrm{d} = 1$, is strictly concave for $-2 \leq s < 2$, affine linear for $s = 2$, and strictly convex for $s > 2$.

Remark: Restricted to $s \in \{-2, -1, 0\}$ the concavity as stated in the Conjecture has been proven, as stated earlier. The affine linearity for $s = 2$ and the strict convexity for when $s = 2m$, $m \in \mathbb{N}$, one can prove to be a corollary to a theorem by Johann Brauchart, see below. A proof of the complete Conjecture 1 would be accomplished if one could show that for each $N \geq 3$ the continuous function $s \mapsto \ddot{v}_s(N)$ is (strictly) negative for $s < 2$ and (strictly) positive for $s > 2$. MAPLE plots of $s \mapsto \ddot{v}_s(N)$ for $N \in \{3, ..., 10\}$ reinforce the conjecture.

Further numerical evidence for Conjecture 1



The second discrete derivative $\ddot{v}_s(N)$ as function of s for $N \in \{3,..., 10\}$ seems to be negative for $s < 2$ and positive for $s > 2$.

A MAPLE results-inspired second Conjecture and a related Theorem: first evidence ...

MAPLE-generated plots of $N \mapsto v_s(N)$ for $s \in \{1, 2, 3, 4, 5, 6\}$ suggested that $N \mapsto v_2(N)$ was linear in N.
This was reinforced by the MAPLE-generated plots of $N \mapsto \ddot{v}_s(N)$ for this selection of s-values, which produced an optically vanishing sequence $N \mapsto \ddot{v}_2(N)$.
MAPLE's algebraic **eval** routine reinforced this by producing

$$v_2(N+1) - v_2(N) = \frac{1}{24}$$

and

$$v_2(N) - \frac{N}{24} = -\frac{11}{24}$$

for $2 \leq N \leq 10$ (computing "forever" when $N > 10$), suggesting

$$\boxed{v_2(N) = \frac{1}{24}\left(-11 + N\right)}.$$

A MAPLE results-inspired second Conjecture and a related Theorem: further evidence ...

The MAPLE-generated plots of $N \mapsto \ddot{v}_s(N)$ for $s \in \{1, 2, 3, 4, 5, 6\}$ suggested that $N \mapsto \ddot{v}_4(N)$ was linear in N. This was reinforced by the MAPLE-generated plots of $N \mapsto \ddddot{v}_s(N)$ for this selection of s-values, which produced an optically vanishing sequence $N \mapsto \ddddot{v}_4(N)$.
MAPLE's algebraic **eval** routine reinforced this by producing

$$\ddot{v}_4(N+1) - \ddot{v}_4(N) = \tfrac{1}{480}$$

and

$$\ddot{v}_4(N) - \tfrac{N}{480} = \tfrac{1}{1440}$$

for $3 \leq N \leq 5$ (computing forever when $N > 5$ ...), suggesting

$$\boxed{\ddot{v}_4(N) = \tfrac{1}{1440}\,(1+3N)}.$$

A MAPLE results-inspired second Conjecture and a related Theorem: The Conjecture ...

When empirically also the 4th discrete derivative of $N \mapsto v_6(N)$ and the 6th discrete derivative of $N \mapsto v_8(N)$ turned out to be affine linear, the following conjecture was in order:

Conjecture 2: Whenever $s = 2m$ with $m \in \mathbb{N}$, then the average s-Riesz pair energy of the roots of unity, i.e. $N \mapsto v_s(N)$ when $\mathrm{d} = 1$, is a polynomial in N of degree $2m - 1$.

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Remark: While collaborating with Rachele Nerrattini and Johann Brauchart on our JSP paper, I communicated Conjecture 2 to them. I was amazed when Johann wrote back that he had recently proved a related theorem, inspired by results proved earlier in his joint work with Hardin and Saff. In fact, Johann has proved two theorems, one for negative even s values, one for positive even s values.

A theorem of Johann Brauchart for negative even s

Theorem 1: *For $m \in \mathbb{N}$ one has*

$$\sum_{k=1}^{N-1} \sin^{2m}\left(\frac{k\pi}{N}\right) = \frac{1}{2^{2m}}\left(\binom{2m}{m} + 2\sum_{\substack{k=1\\ N|k}}^{m}(-1)^k\binom{2m}{m-k}\right)N$$

The sum over k includes only those k that are divided by N, and so it vanishes if $N > m$.

Remark: Brauchart's short proof is by direct computation, based on standard trigonometric identities and known finite sums of sine and cosine. Note that the expression at r.h.s. is generally piecewise linear in N. Theorem 1 is inspired by earlier results of Brauchart, Hardin, and Saff, to which I come later.

A theorem of Johann Brauchart for positive even s

Theorem 2: *Set $s = 2m$ with $m \in \mathbb{N}$. Then*

$$\sum_{k=1}^{N-1} \sin^{-2m}\left(\frac{k\pi}{N}\right) = \frac{2}{\pi^{2m}} \sum_{k=0}^{m} \alpha_k(2m)\zeta\big(2(m-k)\big) N^{2(m-k)}.$$

Here, the coefficients $\alpha_k(2m) > 0$ are defined by the generating function

$$\left(\frac{\pi z}{\sin(\pi z)}\right)^{2m} = \sum_{k=0}^{\infty} \alpha_k(2m) z^{2k}; \quad |z| < 1. \qquad (1)$$

Remark: Brauchart's (not so short) proof is by direct **massive**(!) computation. Theorem 2 is inspired by earlier results of Brauchart, Hardin, and Saff, to which I come later.

Direct spin-off of Brauchart's theorem for even $s > 0$

Corollary: For $s = 2m$, $m \in \mathbb{N}$, the Riesz s-energy, per root, of the N-th root of unity, well-defined for $N \geq 2$, and given by

$$\frac{1}{4m}\left(1 - N + \sum_{k=1}^{N-1} \frac{1}{2^{2m}\sin^{2m}\left(\frac{k\pi}{N}\right)}\right)$$

$$= \frac{1}{4m}\left(1 - N + \frac{2}{(2\pi)^{2m}} \sum_{k=0}^{m} \alpha_k(2m)\zeta\big(2(m-k)\big) N^{2(m-k)}\right),$$

is strictly convex at each $N \geq 3$.

Remark: Brauchart, Hardin, and Saff observed that for $m \in \mathbb{N}$ the coefficients $\alpha_k(2m)\zeta(2m-2k)$ of $N^{2(m-k)}$ in the above polynomials are strictly positive when $k < m$ and strictly negative only when $k = m$. The linear term is negatively proportional to N, but that does not affect the curvature. Strict convexity follows.

Special cases of Brauchart's theorem for even $s > 0$

$s = 2$: $\sum_{k=1}^{N-1} \sin^{-2}\left(\frac{k}{N}\pi\right) = \frac{1}{3}(N-1)(N+1)$

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We see that when $s \in \{2, 4, 6\}$, then $\sum_{k=1}^{N-1} \sin^{-s}\left(\frac{k}{N}\pi\right)$ [a-priori well-defined only for $N \geq 2$] is a polynomial in N that factors into a product of $(N-1)$ times a convex polynomial in N. Thus, extended (as a polynomial) to $N \geq 1$, it **vanishes** for $N = 1$.

This observation generalizes to all even positive s values!

A vanishing identity

Conjecture: For all $m \in \mathbb{N}$, with $\alpha_k(2m)$ as defined earlier,

$$\boldsymbol{\Sigma}(m) := \tfrac{2}{\pi^{2m}} \sum_{k=0}^{m} \alpha_k(2m)\zeta\big(2(m-k)\big) = 0.$$

We first give a physicist's "proof," that leads to this conclusion for the wrong reasons, and then a mathematician's argument that suggests this conclusion for the right reasons, and that I expect can be proved without too much additional effort.

Note added: Shortly after this presentation, the proof of the above conjecture was completed thanks to an observation by Doron Zeilberger. See the slides added after the bibliography.

How not to prove the vanishing identity

Physicist's "Proof": By Brauchart's Theorem 2, we know that for all integers $N \geq 2$ and $m \in \mathbb{N}$ we have

$$\frac{2}{\pi^{2m}} \sum_{k=0}^{m} \alpha_k(2m)\zeta\big(2(m-k)\big) N^{2(m-k)} = \sum_{k=1}^{N-1} \sin^{-2m}\big(\tfrac{k\pi}{N}\big).$$

At l.h.s. we can set $N = 1$, so it must be OK at r.h.s., viz.

$$\mathbf{\Sigma}(m) = \textstyle\sum_{k=1}^{0} \sin^{-2m}(k\pi).$$

Now $\sum_{k=1}^{0}(\cdots)_k \equiv 0$ by definition, and this makes sense from a physics point of view, for when $N = 1$ there are no other points the given point can interact with, so its interaction energy must vanish. Q.E.D.

How to prove the vanishing identity

Mathematical Evidence: We recall that Brauchart, Hardin, and Saff defined the $\alpha_k(2m)$ in terms of a generating function that is the negative $2m$-th power of sinc(πz); hence, recalling also the known values of the ζ function at even integers, we can write

$$\alpha_k(2m)\zeta\big(2(m-k)\big) = \frac{1}{(2k)!}\frac{d^{2k}}{dz^{2k}}\left(\frac{\pi z}{\sin(\pi z)}\right)^{2m}\Big|_{z=0}\times$$
$$\times(-1)^{m-k+1}\frac{(2\pi)^{2(m-k)}}{2\big(2(m-k)\big)!}B_{2(m-k)},$$

where $B_{2(m-k)}$ is the $2(m-k)$-th Bernoulli number,

$$B_{2(m-k)} := \frac{d^{2(m-k)}}{dz^{2(m-k)}}\left(\frac{z}{e^z-1}\right)\Big|_{z=0};$$

note that this includes the case $k=m$ with $\zeta(0)=-\frac{1}{2}$.
After some cancellations and variable substitutions we find

How to prove the vanishing identity

$$\mathbf{\Sigma}(m) =$$

$$(-1)^{m+1}\frac{2^{2m}}{(2m)!}\sum_{k=0}^{m}\frac{(-1)^k}{2^{2k}}\binom{2m}{2k}\left[\frac{d^{2k}}{dz^{2k}}\frac{z^{2m}}{\sin^{2m}(z)}\right]\left[\frac{d^{2(m-k)}}{dz^{2(m-k)}}\frac{z}{e^z-1}\right]\Big|_{z=0}$$

$$=(-1)^{m+1}\frac{2^{2m}}{(2m)!}\sum_{k=0}^{m}\binom{2m}{2k}\left[\frac{d^{2k}}{dz^{2k}}\frac{(z/2)^{2m}}{\sinh^{2m}\left(\frac{z}{2}\right)}\right]\left[\frac{d^{2(m-k)}}{dz^{2(m-k)}}\frac{z}{e^z-1}\right]\Big|_{z=0}$$

$$=(-1)^{m+1}\frac{2^{2m}}{(2m)!}\sum_{k=0}^{2m}\binom{2m}{k}\left[\frac{d^{k}}{dz^{k}}\frac{(z/2)^{2m}}{\sinh^{2m}\left(\frac{z}{2}\right)}\right]\left[\frac{d^{2m-k}}{dz^{2m-k}}\frac{z}{e^z-1}\right]\Big|_{z=0}$$

$$=(-1)^{m+1}\frac{2^{2m}}{(2m)!}\frac{d^{2m}}{dz^{2m}}\left[\frac{(z/2)^{2m}}{\sinh^{2m}\left(\frac{z}{2}\right)}\frac{z}{e^z-1}\right]\Big|_{z=0}$$

$$=(-1)^{m+1}\frac{2^{2m}}{(2m)!}\frac{d^{2m}}{dz^{2m}}\left[\left(\frac{z}{e^z-1}\right)^{2m+1}e^{mz}\right]\Big|_{z=0}$$

$$=(-1)^{m+1}\frac{2^{2m}}{(2m)!}B_{2m}^{[2m+1]}(m) = 0 \text{ for all} m \in \mathbb{N}(\text{well}, m \leq 20+\ldots)\text{“Q.E.D}$$

Convexity of $N \mapsto \langle V_{2m} \rangle \left(\omega_1^{(N)}\right)$, $m \in \mathbb{N}$

"**Experimental Theorem**": $N = 1$ is always a zero of the strictly convex polynomial in N that, for $N \geq 2$ and $m \in \mathbb{N}$, is identical to $\sum_{k=1}^{N-1} \sin^{-2m}\left(\frac{k\pi}{N}\right)$. As a consequence, when $\mathrm{d} = 1$ then $N \mapsto v_{2m}(N)$ is convex at each $N \geq 3$ and $m \in \mathbb{N}$.

N.B.: Shortly after the presentation, this "experimental theorem" had become a proper theorem; see the slides added after the bibliography.

Some examples of $N \mapsto \langle V_{2m} \rangle \left(\omega_1^{(N)}\right)$, $m \in \mathbb{N}$

For $\mathrm{d} = 1$ we have, when $s \in \{2, 4, 6\}$:

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The result for $s = 2$ proves that $N \mapsto v_2(N)$ is affine linear, as conjectured based on our empirical MAPLE study. The results for $s \in \{4, 6\}$ prove that $N \mapsto v_4(N)$ and $N \mapsto v_6(N)$ are strictly convex, as also conjectured. Empirically (and shortly after the presentation: rigorously):
When $m > 1$, then $\sum_{k=1}^{N-1} \sin^{-2m}\left(\frac{k\pi}{N}\right)$ (well-defined for $N \geq 2$) is a polynomial in N that always factorizes into a product of $(N-1)$ and a strictly convex polynomial. Hence, in $\mathrm{d} = 1$:
strict convexity of $N \mapsto v_{2m}(N)$ for $m \in \mathbb{N}$ at $N \geq 3$, follows.

A cautionary tale for experimental mathematicians ...

Johann Brauchart's two theorems were inspired by earlier results of Brauchart, Hardin, Saff on **the complete asymptotic large-N expansion** of (in my idiosyncratic notion)

$$S(N;s) := \sum_{k=1}^{N-1} \sin^{-s}\left(\frac{k\pi}{N}\right).$$

They proved the following
Theorem: $\forall n \in \mathbb{N}$ and $\forall s \in \mathbb{C}\setminus\{0,1,3,5,...\}$, as $N \to \infty$,

$$S(N;s) = R_s N + N^s \frac{2}{\pi^s} \sum_{k=0}^{n} \alpha_k(s)\zeta(s-2k)N^{-2k} + \mathcal{O}_{s,n}\left(N^{\Re s-2(n+1)}\right)$$

where

$$R_s = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)}$$

and $\mathcal{O}_{s,n}(X)$ means terms bounded by $C(\Re s, n)N^{\Re s-2(n+1)}$.

A cautionary tale for experimental mathematicians ...

Brauchart, Hardin, and Saff remarked that when s is a non-zero even integer, then $\zeta(s-2n)=0$ whenever $2n>s$, and then the a-priori arbitrarily long but finite asymptotic expansion (finite up to an error term depending on the length of the expansion) terminates after at most $s/2$ terms, no matter how high the asymptotics is pushed in n.

Moreover, when $s=2m$, $m\in\mathbb{N}$, then $R_s=0$, for $\frac{1}{\Gamma(1-\frac{s}{2})}=0$.

This suggests that $S(N;\pm 2m)$, $m\in\mathbb{N}$, is a polynomial.

Brauchart's theorems reveal:

True for $s=2m$ and $s=-2$, but false when $s=-2m$, $m>1$.

Things to try for experimentalists:

1. Brauchart's proof of his theorem for even $s > 0$ is a **tour de force**. But the few explicitly computed polynomials look simple, with rational coefficients. All factors with powers of π coming from ζ and α cancelled. Is there a computer algebraic proof of the theorem? — And better yet: Is there a proof that produces simple closed form expressions for the coefficients ?

Things to try for experimentalists:

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Things to try for experimentalists:

1. Brauchart's proof of his theorem for even $s > 0$ is a **tour de force.** But the few explicitly computed polynomials look simple, with rational coefficients. All factors with powers of π coming from ζ and α cancelled. Is there a computer algebraic proof of the theorem? — And better yet: Is there a proof that produces simple closed form expressions for the coefficients ?
2. Are there novel integer sequences associated with these polynomials?
3. What when s is an odd positive integer? In that case the complete asymptotic expansion of Brauchart, Hardin, and Saff for $\sum_{k=1}^{N-1} \sin^{-s}\left(\frac{k\pi}{N}\right)$ does not terminate after a finite number of terms, so one will not obtain a polynomial. Are there closed form expressions, such as when $s = -1$?

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- ALL OF YOU FOR ATTENDING!

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Added Feb.17, 2022, after the presentation

Completion of the proof of the vanishing identity

After my presentation Doron Zeilberger remarked that the monomial factor z^{2m+1} in the generating function of the $2m$-th generalized Bernoulli polynomial of order $2m+1$, viz. $B_{2m}^{[2m+1]}(x)$, cancels under the integral of a Cauchy residue representation of that polynomial, and that MAPLE would recognize the remaining integrand as a total derivative, as a result of which the remaining step of the proof of the "vanishing-identity conjecture" is completed.
After rewriting the generating function in terms of the monomial in z and otherwise only hyperbolic sine and cosine functions, this argument simplifies to the point that MAPLE is not needed.

Proof of the vanishing identity — completed

Thus,

$$B_{2m}^{[2m+1]}(m) = \frac{d^{2m}}{dz^{2m}}\left[\left(\frac{z}{e^z-1}\right)^{2m+1} e^{mz}\right]\Big|_{z=0}$$

$$= \frac{1}{2^{2m}}\frac{d^{2m}}{dz^{2m}}\left[\frac{z^{2m+1}}{\sinh^{2m+1}(z)}\cosh(z)\right]\Big|_{z=0}$$

$$= \frac{1}{2^{2m}}\frac{(2m)!}{2\pi i}\int_{\mathbb{S}^1}\frac{\cosh\zeta}{(\sinh\zeta)^{2m+1}}d\zeta$$

$$= \frac{1}{2^{2m}}\frac{(2m)!}{2\pi i}\int_{\sinh\mathbb{S}^1}\frac{1}{w^{2m+1}}dw$$

Clearly, $\frac{1}{w^{2m+1}}dw = -\frac{1}{2m}d\frac{1}{w^{2m}}$, and so the integral vanishes.
Q.E.D.