GUESSING WITH LITTLE DATA

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3, 5, 7, 9, 11, 13, ?, ?, ?, ?, ?, ?

$3,\ 5,\ 7,\ 9,\ 11,\ 13,\ \pi,\ \mathrm{e},\ \sqrt{2},\ \zeta(3),\ \log(2),\ \mathrm{i}$

3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25

3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25

1, 3, 9, 21, 41, 71, ?, ?, ?, ?, ?, ?

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$$\underbrace{1,\ 3,\ 9,\ 21,\ 41,\ 71,}_{\downarrow\ \text{interpolate}}\ ?,\ ?,\ ?,\ ?,\ ?,\ ?$$

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$$\underbrace{1,\ 3,\ 9,\ 21,\ 41,\ 71,}_{\substack{\downarrow\ \text{interpolate} \\ \frac{1}{3}(x^3 - x) + 1}}\ 113,\ 169,\ 241,\ 331,\ 441,\ 573$$

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1, 5, 19, 65, 211, 665, ?, ?, ?, ?, ?

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$$\frac{1}{60}\left(47x^5 - 590x^4 + 3065x^3 - 7570x^2 + 8888x - 3780\right)$$

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$$\underbrace{1,\ 5,\ 19,\ 65,\ 211,\ 665,}_{}\ 2059,\ 6305,\ 19171,\ 58025,\ 175099$$

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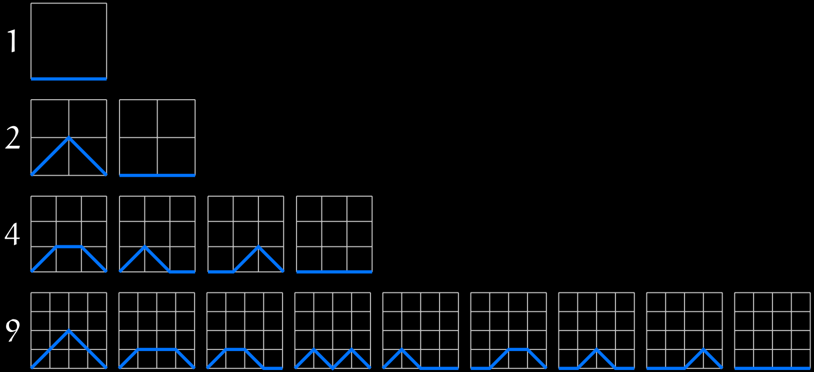
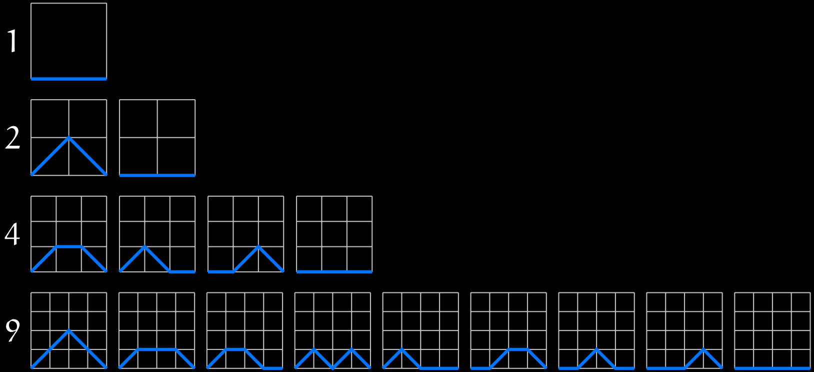
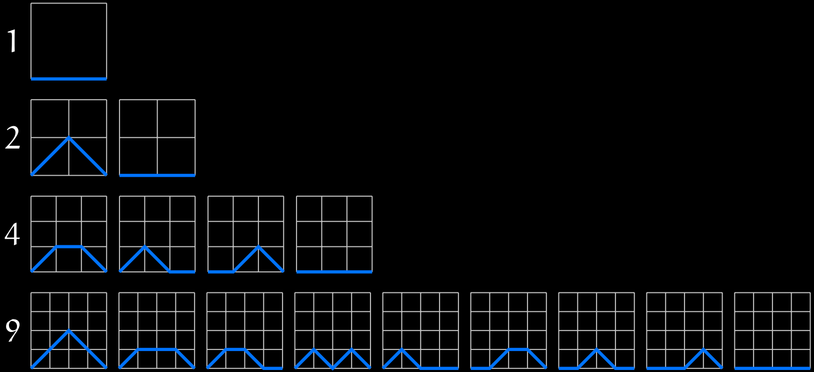
- $(3n^2+5n-7)a_{n+2}+(5n^2+2n+8)a_{n+1}+(2n^2-9n-1)a_n = 0$


1
2
4
9



21, 51, 127, 323, 835, 2188, …



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Is this sequence D-finite?

Let's try to find a recurrence (candidate)!

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$$\begin{aligned} &(c_{2,0} + c_{2,1}n + c_{2,2}n^2)a_{n+2} \\ +&(c_{1,0} + c_{1,1}n + c_{1,2}n^2)a_{n+1} \\ +&(c_{0,0} + c_{0,1}n + c_{0,2}n^2)a_n = 0 \end{aligned}$$

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Instantiate this template ("ansatz") for $n = 1, \ldots, 8$ to obtain a system of linear constraints for the undetermined coefficients.

$$\begin{array}{cc} & \begin{array}{ccccccccc} a_n & na_n & n^2a_n & a_{n+1} & na_{n+1} & n^2a_{n+1} & a_{n+2} & na_{n+2} & n^2a_{n+2} \end{array} \\ \begin{array}{c} n{=}1 \\ n{=}2 \\ n{=}3 \\ n{=}4 \\ n{=}5 \\ n{=}6 \\ n{=}7 \\ n{=}8 \end{array} & \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 4 & 4 & 4 \\ 2 & 4 & 8 & 4 & 8 & 16 & 9 & 18 & 36 \\ 4 & 12 & 36 & 9 & 27 & 81 & 21 & 63 & 189 \\ 9 & 36 & 144 & 21 & 84 & 336 & 51 & 204 & 816 \\ 21 & 105 & 525 & 51 & 255 & 1275 & 127 & 635 & 3175 \\ 51 & 306 & 1836 & 127 & 762 & 4572 & 323 & 1938 & 11628 \\ 127 & 889 & 6223 & 323 & 2261 & 15827 & 835 & 5845 & 40915 \\ 323 & 2584 & 20672 & 835 & 6680 & 53440 & 2188 & 17504 & 140032 \end{pmatrix} \end{array} \begin{pmatrix} c_{0,0} \\ c_{0,1} \\ c_{0,2} \\ c_{1,0} \\ c_{1,1} \\ c_{1,2} \\ c_{2,0} \\ c_{2,1} \\ c_{2,2} \end{pmatrix} = 0$$

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More variables than equations.

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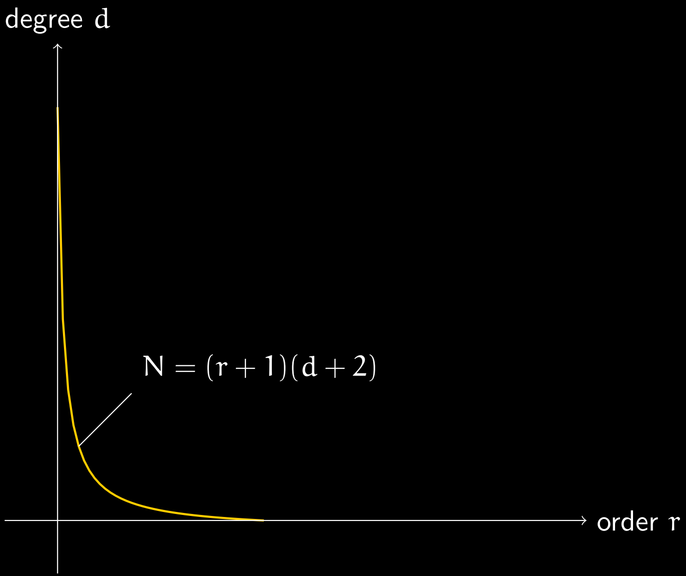
This inequality determines the boundary of the search space.


degree d
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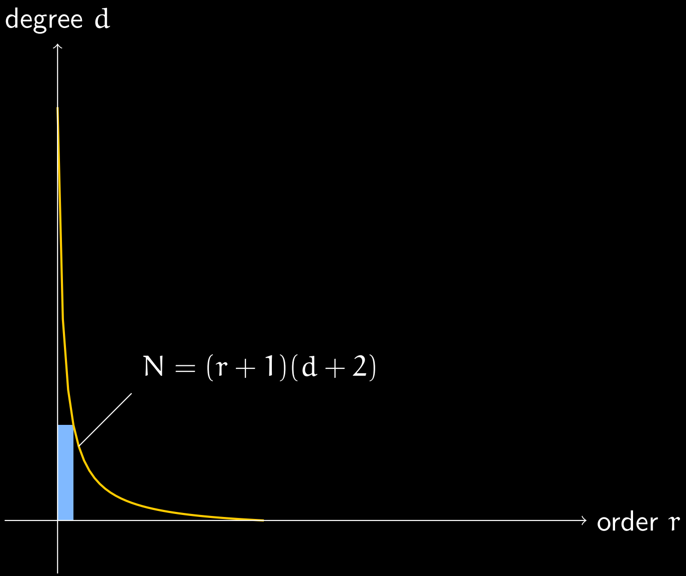

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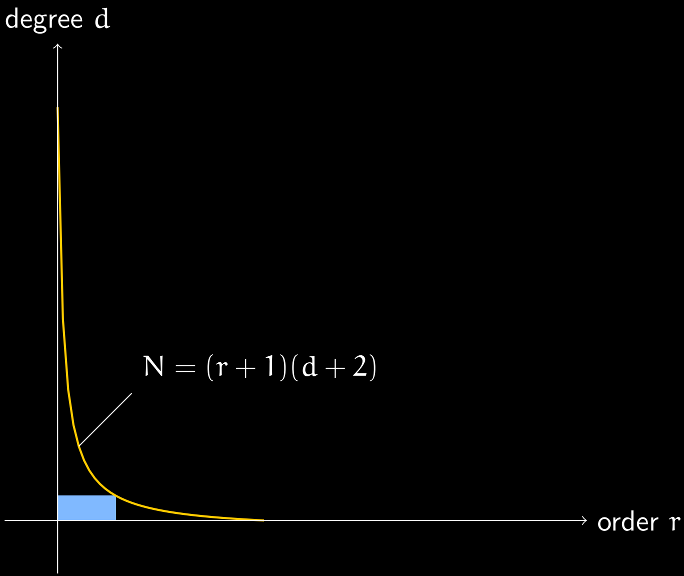

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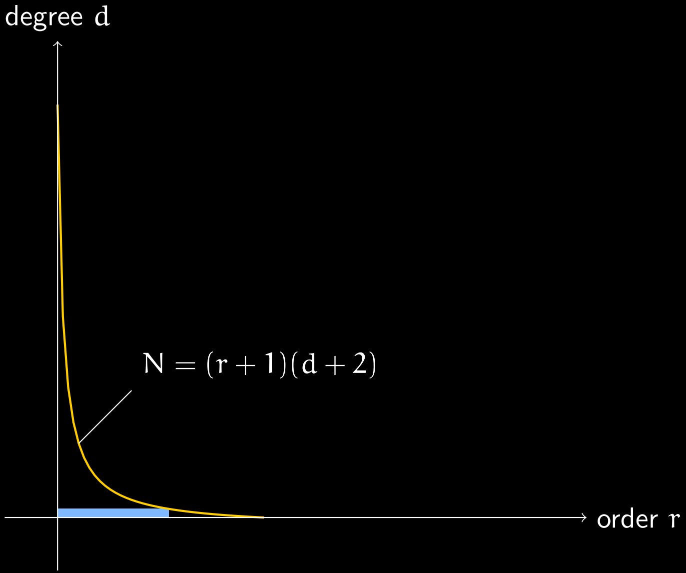
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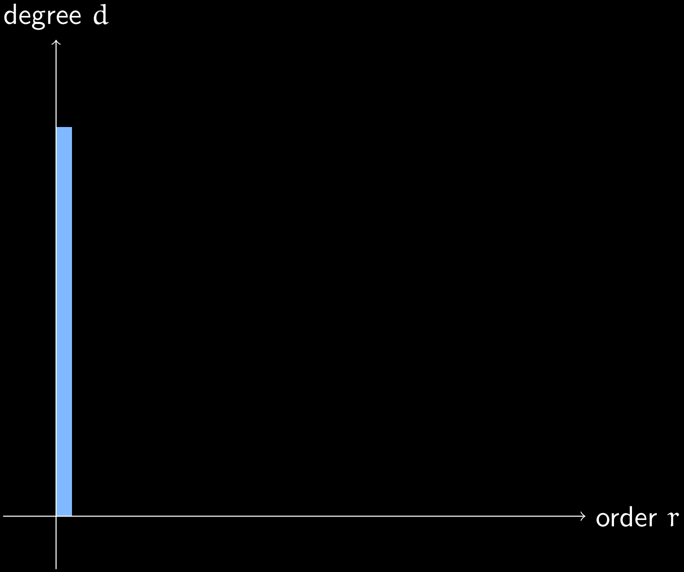
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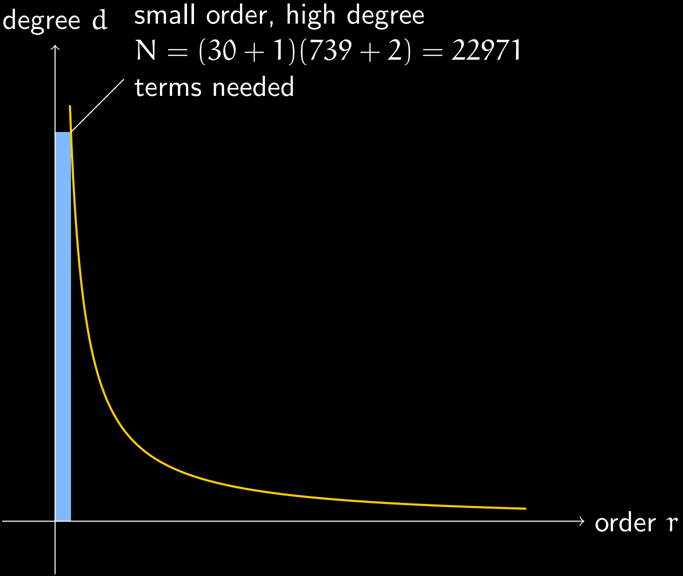
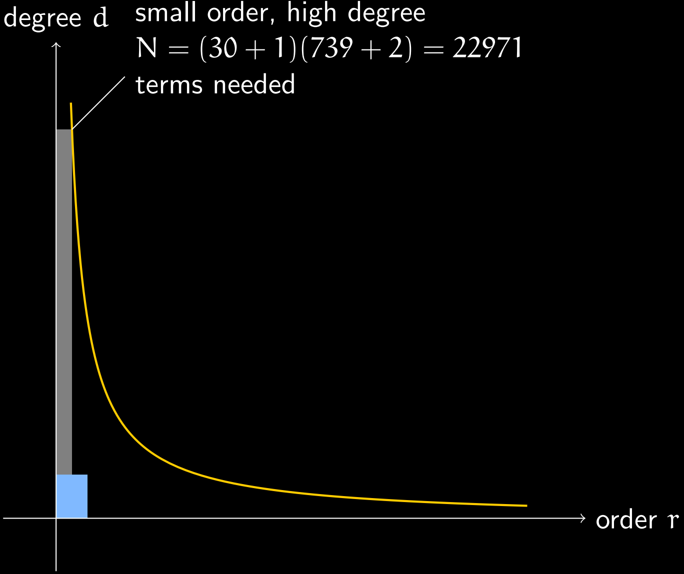


small order, high degree

$N = (30+1)(739+2) = 22971$

terms needed



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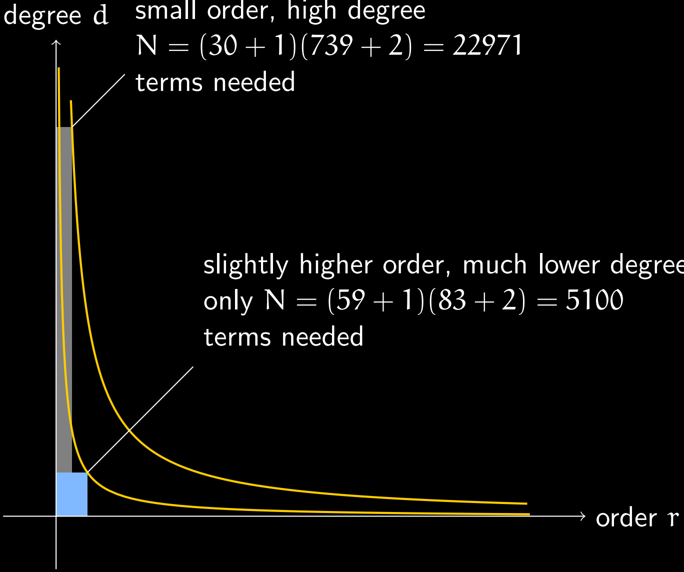
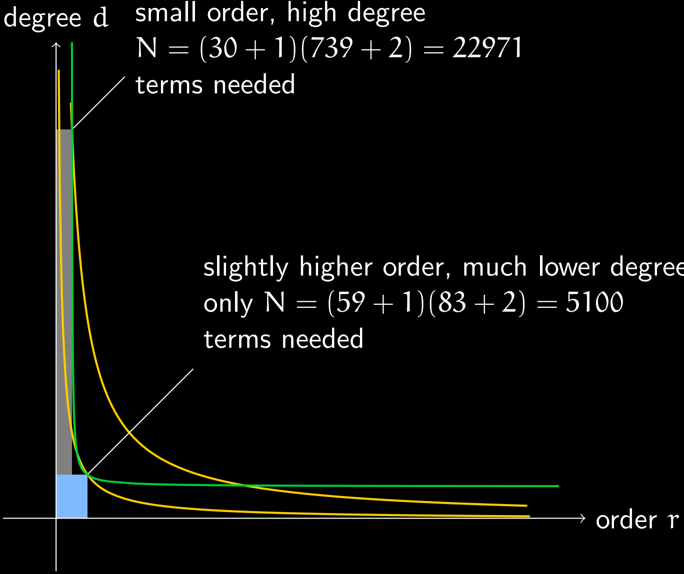
terms needed



small order, high degree
$N = (30+1)(739+2) = 22971$
terms needed

slightly higher order, much lower degree
only $N = (59+1)(83+2) = 5100$
terms needed



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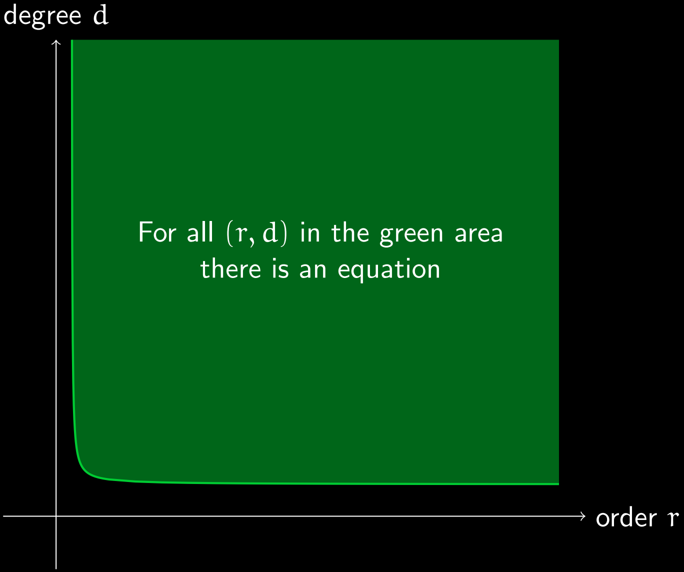

For all (r, d) in the green area
there is an equation

	minimal order	non-minimal order
degree	very high	better
integer lengths	better	very long

	minimal order	non-minimal order
degree	very high	better
integer lengths	better	very long

We can efficiently get the minimal-order equation from a non-minimal-order equation.

What if we don't have enough data?

- For large examples, there is an old trick.
- For small examples, there is a new trick.

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- For small examples, there is a new trick.

Guessing with Little Data[*]

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ABSTRACT

Reconstructing a hypothetical recurrence equation from the first terms of an infinite sequence is a classical and well-known technique in experimental mathematics. We propose a variation of this technique which can succeed with fewer input terms.

CCS CONCEPTS

• **Computing methodologies → Algebraic algorithms**.

KEYWORDS

Experimental Mathematics, D-finite Functions, Lattice Reduction, Integer Sequences

ACM Reference Format:
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1 INTRODUCTION

A simple but powerful technique which has become an important tool in experimental mathematics takes as input the first few terms of an infinite sequence and returns as output a plausible hypothesis for a recurrence equation that the sequence may satisfy, or a plausible hypothesis for a differential equation satisfied by its generating function. The principle is known as automated guessing as it somehow makes a guess how the infinite sequence continues beyond the finitely many terms supplied as input. In certain situations where sufficient additional information is available about the sequence at hand, automated guessing can be combined with other techniques from computer algebra that confirm that the guessed equation is correct. One of many successful applications of this paradigm is the proof of the qTSPP conjecture [21].

Technically, the guessing problem for linear recurrences can be solved by linear algebra. Given $a_0, \ldots, a_N$, we choose an order r and a degree d and make an ansatz with undetermined coefficients: $\sum_{i=0}^{r} \sum_{j=0}^{d} c_{i,j} n^j a_{n+i} = 0$. Using the available terms of the sequence, we can instantiate the ansatz for $n = 0, \ldots, N - r$ and get $N - r + 1$ linear constraints on the $(r+1)(d+1)$ unknown coefficients $c_{i,j}$. If r and d are chosen such that $(r+1)(d+2) \leq N+2$, this homogeneous linear system is not expected to have a (nontrivial) solution. If it does have a solution, this is interpreted as evidence in favor of the correctness of the corresponding equation. The more the number of equations exceeds the number of variables, the stronger is the evidence that the solution is not just noise but means something.

The linear systems arising from guessing problems can be solved efficiently by Hermite-Padé approximation [1, 8, 13, 15]. Modern implementations have no trouble handling examples where N is in the range of 10000. There are also variants for the multivariate setting [2–4] as well as experiments with approaches that use machine learning instead of linear algebra [12].

Although it rarely happens in practice, a guessed equation may be incorrect. It is therefore important to be able to assess the quality of a guess. The amount of overdetermination of the linear system was already mentioned as a source of trust. Some further tests that may help to distinguish correct equations from noise have been proposed in [9]: a correct equation is likely to contain short integer coefficients while a wrong guess will typically contain long integers; a correct recurrence for an integer sequence must produce only integers when it is unrolled while a wrong guess will typically produce rational numbers; a correct differential equation for a generating function is likely to have nice singularities while a wrong guess will typically have awkward singularities; a correct differential equation for a generating function of an integer sequence with moderate growth must have nilpotent p-curvature [6, 7, 24] while a wrong guess will typically not have this property. All these tests are extremely strong and make guessing a very reliable tool in practice. Especially for large examples we virtually never encounter wrong guesses.

The focus of this paper is on small examples. The assumption is that N is small and that further terms cannot be obtained at reasonable cost. A prominent example for such a situation is the number of permutations avoiding the pattern 1324 for which, despite tremendous efforts [11, 16], only the first 50 terms are known. If such a sequence satisfies an equation of order r and degree d with $(r+1)(d+2) > N+2$, we won't be able to find this equation directly. In this situation, it can be exploited that an equation of slightly higher order may have substantially lower degree, so that we may find an equation using $r+1$ and $d/2$, for example. This phenomenon is well understood [14] and has been exploited by guessing software since long. Our assumption here is that N is so small that for every choice (r, d) with $(r+1)(d+2) \leq N+2$ the linear system has no solution, so that trading order against degree does not help. For sequences where every other term is zero it has been observed [19] that removing the zeros from the data can bring

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But the last one is accessible via LLL ☺

$a_n = \sum_k \binom{n}{k}^3$ satisfies a recurrence of order 2 and degree 2.

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$$\begin{pmatrix} 1 & 0 & 0 & 2 & 0 & 0 & 10 & 0 & 0 \\ 2 & 2 & 2 & 10 & 10 & 10 & 56 & 56 & 56 \\ 10 & 20 & 40 & 56 & 112 & 224 & 346 & 692 & 1384 \\ 56 & 168 & 504 & 346 & 1038 & 3114 & 2252 & 6756 & 20268 \\ 346 & 1384 & 5536 & 2252 & 9008 & 36032 & 15184 & 60736 & 242944 \end{pmatrix} \begin{pmatrix} c_{00} \\ c_{01} \\ c_{02} \\ c_{10} \\ c_{11} \\ c_{12} \\ c_{20} \\ c_{21} \\ c_{22} \end{pmatrix} = 0$$

A basis for the $\mathbb{Z}$-module of all solutions in $\mathbb{Z}^9$ is

$$\begin{pmatrix} 2 \\ 0 \\ 0 \\ 67069 \\ -52693 \\ -45994 \\ -13414 \\ 13424 \\ 5636 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \\ 231310 \\ -181747 \\ -158629 \\ -46262 \\ 46300 \\ 19438 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 2 \\ 232560 \\ -182728 \\ -159486 \\ -46512 \\ 46550 \\ 19543 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 434140 \\ -341119 \\ -297729 \\ -86828 \\ 86900 \\ 36483 \end{pmatrix}$$

A basis for the $\mathbb{Z}$-module of all solutions in $\mathbb{Z}^9$ is

$$\begin{pmatrix} -8 \\ -16 \\ -8 \\ -16 \\ -21 \\ -7 \\ 4 \\ 4 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -8 \\ 21 \\ -11 \\ -6 \\ -29 \\ 1 \\ 2 \\ 4 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 12 \\ 12 \\ -34 \\ -46 \\ 27 \\ 21 \\ 8 \\ -6 \\ -2 \end{pmatrix}, \quad \begin{pmatrix} -42 \\ 14 \\ 58 \\ -29 \\ -17 \\ 40 \\ 10 \\ -4 \\ -6 \end{pmatrix}$$

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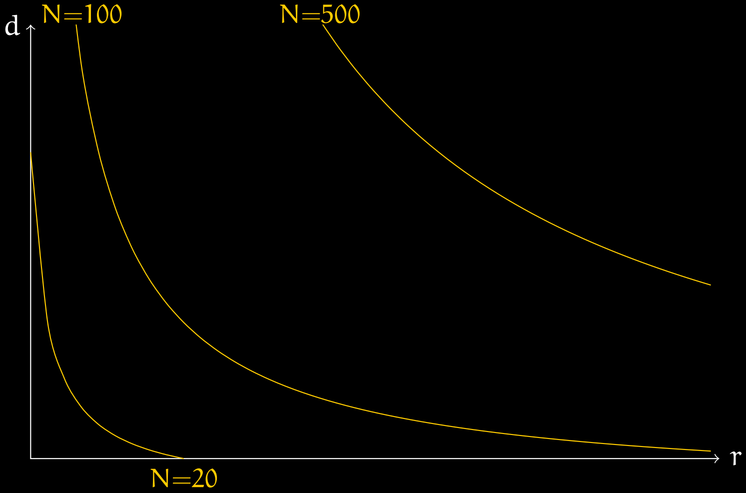
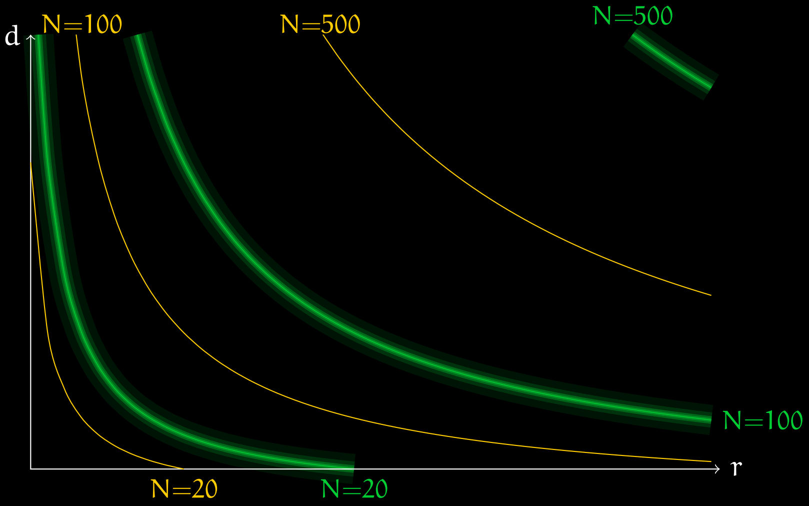
Indeed,

$$(-8-16n-8n^2)a_n+(-16-21n-7n^2)a_{n+1}+(4+4n+n^2)a_{n+2}=0$$

is a correct recurrence.


d
N=100
N=500
r
N=20


N=500
N=100
N=500
d
N=100
r
N=20
N=20

Why don’t we simply compute more terms?

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Because it might be too hard.

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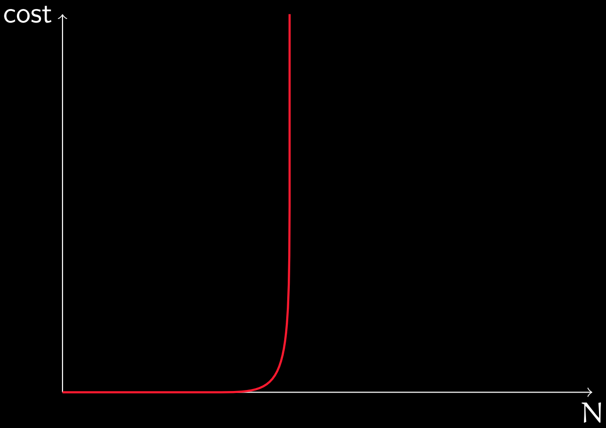

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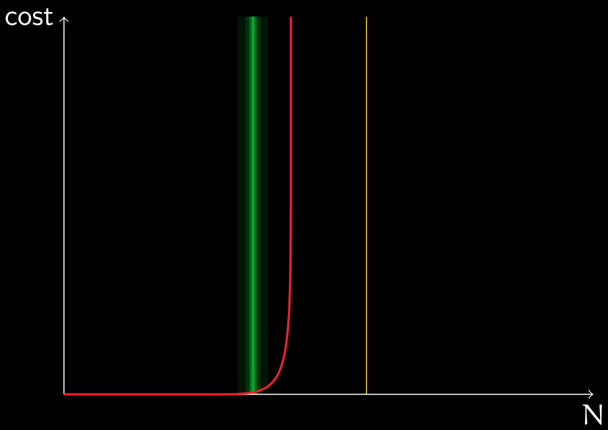

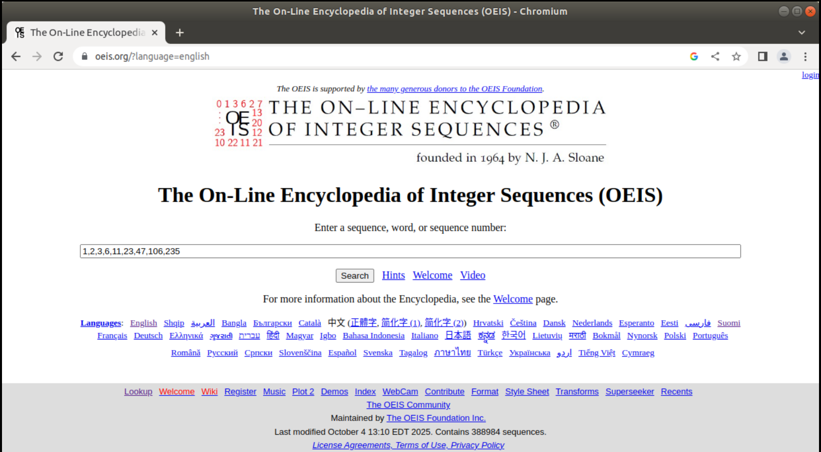

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The On-Line Encyclopedia of Integer Sequences (OEIS)

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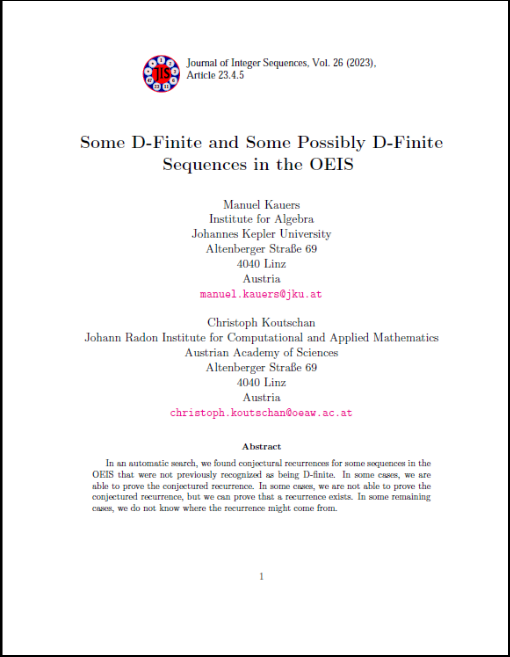

Journal of Integer Sequences, Vol. 26 (2023),
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Some D-Finite and Some Possibly D-Finite Sequences in the OEIS

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Abstract

In an automatic search, we found conjectural recurrences for some sequences in the OEIS that were not previously recognized as being D-finite. In some cases, we are able to prove the conjectured recurrence. In some cases, we are not able to prove the conjectured recurrence, but we can prove that a recurrence exists. In some remaining cases, we do not know where the recurrence might come from.

1

Entry	First terms	N_{oeis}	N_{linalg}	N_{LLL}	r	d
A177317	1, 2, 48, 2288, 135040	29	60	22	3	14
A199250	1, 1, 14, 21, 424, 571	56	98	56	8	18
A250556	8, 60, 302, 1516, 7126	47	58	47	9	8
A265234	1, 43, 2592, 184740	31	56	27	6	6
A172572	90, 67950, 90291600	33	44	17	3	9
A172671	90, 202410, 747558000	33	75	25	4	13
A188818	2, 9, 48, 256, 1360	32	55	26	5	10
A306322	1, 0, 0, 25, 386, 4657	41	63	30	4	14
A195806	16, 105, 496, 1759, 5052	32	41	30	4	10
A216940	260, 27768, 1664244	37	44	29	1	23
A194478	0, 0, 0, 1, 337, 8733	32	35	19	2	12
A215570	1, 35, 18720, 19369350	48	68	27	3	15
A339987	1, 4, 90, 8400, 1426950	40	70	24	5	10
A269021	1, 2, 23, 588, 24553	42	108	28	4	21
A181198	1, 1, 8, 169, 6392	27	33	14	2	9
A181199	1, 1, 16, 985, 141696	26	103	34	3	24
A181280	0, 0, 0, 58, 1629, 28924	27	32	26	10	1
A253217	0, 0, 1, 19, 268, 3568	37	53	27	5	9
A098926	0, 2, 12, 90, 556, 5242	34	55	26	8	7
A164735	0, 0, 0, 0, 0, 0, 0, 1, 0, 4	70	80	66	15	4

Entry	First terms	N_{oeis}	N_{linalg}	N_{LLL}	r	d	
A177317	1, 2, 48, 2288, 135040	29	60	22	3	14	✓
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A195806	16, 105, 496, 1759, 5052	32	41	30	4	10	✓
A216940	260, 27768, 1664244	37	44	29	1	23	✓
A194478	0, 0, 0, 1, 337, 8733	32	35	19	2	12	✓
A215570	1, 35, 18720, 19369350	48	68	27	3	15	?
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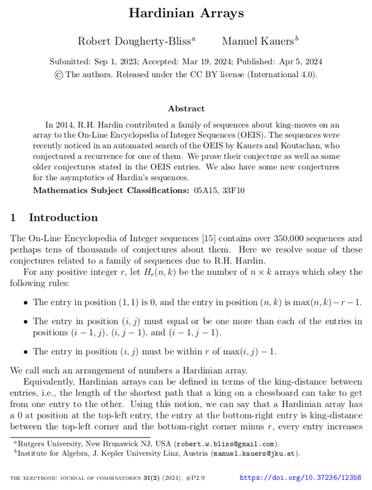
A181199	1, 1, 16, 985, 141696	26	103	34	3	24	?
A181280	0, 0, 0, 58, 1629, 28924	27	32	26	10	1	✓
A253217	0, 0, 1, 19, 268, 3568	37	53	27	5	9	✓
A098926	0, 2, 12, 90, 556, 5242	34	55	26	8	7	?
A164735	0, 0, 0, 0, 0, 0, 0, 1, 0, 4	70	80	66	15	4	?



Hardinian Arrays

Robert Dougherty-Bliss[a]   Manuel Kauers[b]

Submitted: Sep 1, 2023; Accepted: Mar 19, 2024; Published: Apr 5, 2024

Abstract

In 2014, R.H. Hardin contributed a family of sequences about king-moves on an array to the On-Line Encyclopedia of Integer Sequences (OEIS). The sequences were recently noticed in an automated search of the OEIS by Kauers and Koutschan, who conjectured a recurrence for one of them. We prove their conjecture as well as some older conjectures stated in the OEIS entries. We also have some new conjectures for the asymptotics of Hardin's sequences.

Mathematics Subject Classifications: 05A15, 33F10

1 Introduction

The On-Line Encyclopedia of Integer sequences [15] contains over 350,000 sequences and perhaps tens of thousands of conjectures about them. Here we resolve some of these conjectures related to a family of sequences due to R.H. Hardin.

For any positive integer r, let $H_r(n,k)$ be the number of $n \times k$ arrays which obey the following rules:

- The entry in position $(1,1)$ is 0, and the entry in position (n,k) is $\max(n,k)-r-1$.
- The entry in position (i,j) must equal or be one more than each of the entries in positions $(i-1,j)$, $(i,j-1)$, and $(i-1,j-1)$.
- The entry in position (i,j) must be within r of $\max(i,j)-1$.

We call such an arrangement of numbers a Hardinian array.

Equivalently, Hardinian arrays can be defined in terms of the king-distance between entries, i.e., the length of the shortest path that a king on a chessboard can take to get from one entry to the other. Using this notion, we can say that a Hardinian array has a 0 at position at the top-left entry, the entry at the bottom-right entry is king-distance between the top-left corner and the bottom-right corner minus r, every entry increases

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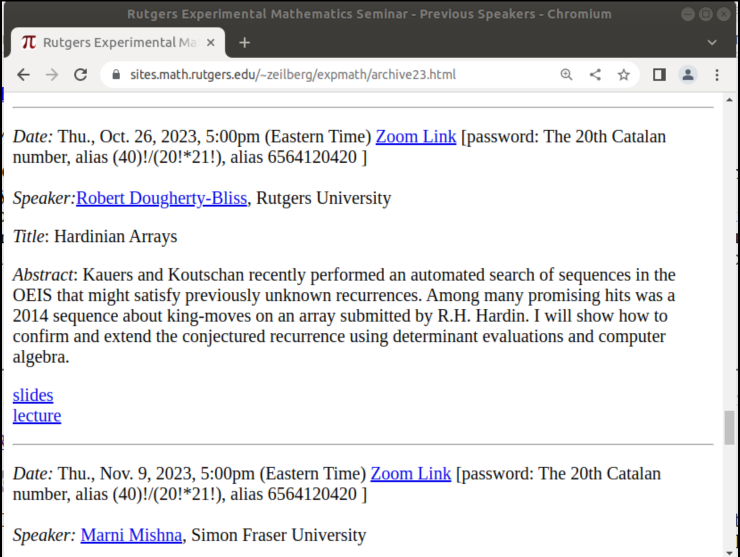
https://doi.org/10.37236/12358



Date: Thu., Oct. 26, 2023, 5:00pm (Eastern Time) Zoom Link [password: The 20th Catalan number, alias (40)!/(20!*21!), alias 6564120420]

*Speaker:*Robert Dougherty-Bliss, Rutgers University

Title: Hardinian Arrays

Abstract: Kauers and Koutschan recently performed an automated search of sequences in the OEIS that might satisfy previously unknown recurrences. Among many promising hits was a 2014 sequence about king-moves on an array submitted by R.H. Hardin. I will show how to confirm and extend the conjectured recurrence using determinant evaluations and computer algebra.

slides
lecture

Date: Thu., Nov. 9, 2023, 5:00pm (Eastern Time) Zoom Link [password: The 20th Catalan number, alias (40)!/(20!*21!), alias 6564120420]

Speaker: Marni Mishna, Simon Fraser University

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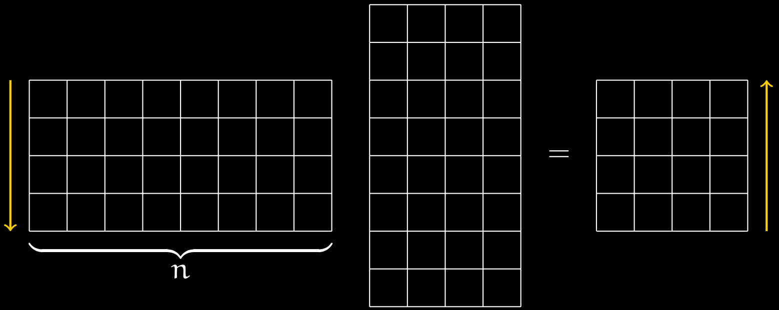
A195806	16, 105, 496, 1759, 5052	32	41	30	4	10	✓
A216940	260, 27768, 1664244	37	44	29	1	23	✓
A194478	0, 0, 0, 1, 337, 8733	32	35	19	2	12	✓
A215570	1, 35, 18720, 19369350	48	68	27	3	15	?
A339987	1, 4, 90, 8400, 1426950	40	70	24	5	10	✓
A269021	1, 2, 23, 588, 24553	42	108	28	4	21	?
A181198	1, 1, 8, 169, 6392	27	33	14	2	9	?
A181199	1, 1, 16, 985, 141696	26	103	34	3	24	?
A181280	0, 0, 0, 58, 1629, 28924	27	32	26	10	1	✓
A253217	0, 0, 1, 19, 268, 3568	37	53	27	5	9	✓
A098926	0, 2, 12, 90, 556, 5242	34	55	26	8	7	?
A164735	0, 0, 0, 0, 0, 0, 0, 1, 0, 4	70	80	66	15	4	?

Entry	First terms	N_{oeis}	N_{linalg}	N_{LLL}	r	d	
A177317	1, 2, 48, 2288, 135040	29	60	22	3	14	✓
A199250	1, 1, 14, 21, 424, 571	56	98	56	8	18	✓
A250556	8, 60, 302, 1516, 7126	47	58	47	9	8	✓
A265234	1, 43, 2592, 184740	31	56	27	6	6	✓
A172572	90, 67950, 90291600	33	44	17	3	9	✓
A172671	90, 202410, 747558000	33	75	25	4	13	✓
A188818	2, 9, 48, 256, 1360	32	55	26	5	10	✓
A306322	1, 0, 0, 25, 386, 4657	41	63	30	4	14	✓
A195806	16, 105, 496, 1759, 5052	32	41	30	4	10	✓
A216940	260, 27768, 1664244	37	44	29	1	23	✓
A194478	0, 0, 0, 1, 337, 8733	32	35	19	2	12	✓
A215570	1, 35, 18720, 19369350	48	68	27	3	15	?
A339987	1, 4, 90, 8400, 1426950	40	70	24	5	10	✓
A269021	1, 2, 23, 588, 24553	42	108	28	4	21	?
A181198	1, 1, 8, 169, 6392	27	33	14	2	9	?
A181199	1, 1, 16, 985, 141696	26	103	34	3	24	?
A181280	0, 0, 0, 58, 1629, 28924	27	32	26	10	1	✓
A253217	0, 0, 1, 19, 268, 3568	37	53	27	5	9	✓
A098926	0, 2, 12, 90, 556, 5242	34	55	26	8	7	?
A164735	0, 0, 0, 0, 0, 0, 0, 1, 0, 4	70	80	66	15	4	?

A181280. Number of $4 \times n$ binary matrices M with rows in strictly increasing order and rows of $MM^\top$ (mod 2) in strictly decreasing order.


n
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$$\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&1&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&1&0\\1&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&1&0\\1&1&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&0&1&0\\1&0&0&0\\1&1&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&0&1&0\\1&0&1&0\\1&1&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&0&1&1\\0&1&0&0\\1&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&0&1&1\\0&1&0&0\\1&1&0&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&0&1\\0&0&1&1\\0&1&1&0\\1&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&0&1&1\\0&1&1&0\\1&1&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&0&1&1\\1&0&0&0\\1&1&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&0&1&1\\1&0&1&0\\1&1&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&1&0&0\\0&1&1&0\\1&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&1&0&0\\0&1&1&0\\1&0&1&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&1&0&0\\1&0&0&0\\1&0&1&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&0&1\\0&1&0&1\\0&1&1&0\\1&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&1&0&1\\0&1&1&0\\1&0&1&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&1&0&1\\1&0&0&0\\1&0&1&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&1&1&1\\1&0&0&1\\1&0&1&1\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&1&1&1\\1&0&0&1\\1&1&0&1\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&1&1&1\\1&0&1&1\\1&1&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&1&1&1\\1&0&1&1\\1&1&0&1\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&1&1&1\\1&0&1&1\\1&1&1&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&0&1\\0&1&1&1\\1&1&0&1\\1&1&1&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{pmatrix},\begin{pmatrix}0&0&1&0\\0&0&1&1\\0&1&0&0\\1&0&0&0\end{pmatrix},\begin{pmatrix}0&0&1&0\\0&0&1&1\\0&1&0&0\\1&1&0&0\end{pmatrix},\begin{pmatrix}0&0&1&0\\0&0&1&1\\0&1&0&1\\1&0&0&0\end{pmatrix},\begin{pmatrix}0&0&1&0\\0&0&1&1\\0&1&0&1\\1&1&0&0\end{pmatrix},\begin{pmatrix}0&0&1&0\\0&0&1&1\\1&0&0&0\\1&1&0&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&1&0\\0&0&1&1\\1&0&0&1\\1&1&0&0\end{pmatrix},\begin{pmatrix}0&0&1&0\\0&1&0&0\\0&1&0&1\\1&0&0&0\end{pmatrix},\begin{pmatrix}0&0&1&0\\0&1&0&0\\0&1&0&1\\1&0&0&1\end{pmatrix},\begin{pmatrix}0&0&1&0\\0&1&0&0\\1&0&0&0\\1&0&0&1\end{pmatrix},\begin{pmatrix}0&0&1&0\\0&1&1&0\\1&0&0&0\\1&0&0&1\end{pmatrix},\begin{pmatrix}0&0&1&0\\0&1&1&1\\1&0&1&0\\1&0&1&1\end{pmatrix},\begin{pmatrix}0&0&1&0\\0&1&1&1\\1&0&1&0\\1&1&1&0\end{pmatrix},\begin{pmatrix}0&0&1&0\\0&1&1&1\\1&0&1&1\\1&1&0&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&1&0\\0&1&1&1\\1&0&1&1\\1&1&0&1\end{pmatrix},\begin{pmatrix}0&0&1&0\\0&1&1&1\\1&0&1&1\\1&1&1&0\end{pmatrix},\begin{pmatrix}0&1&0&0\\0&1&0&1\\1&0&0&0\\1&0&1&0\end{pmatrix},\begin{pmatrix}0&1&0&0\\0&1&0&1\\1&0&0&1\\1&0&1&0\end{pmatrix},\begin{pmatrix}0&1&0&0\\0&1&1&0\\1&0&0&0\\1&0&0&1\end{pmatrix},\begin{pmatrix}0&1&0&0\\0&1&1&1\\1&1&0&0\\1&1&0&1\end{pmatrix},\begin{pmatrix}0&1&0&0\\0&1&1&1\\1&1&0&0\\1&1&1&0\end{pmatrix},$$

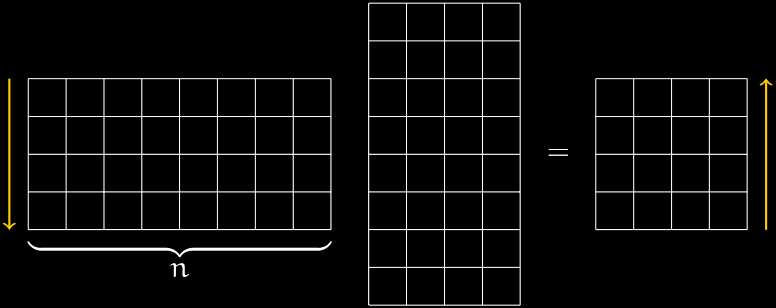
$$\begin{pmatrix}0&1&0&0\\0&1&1&1\\1&1&0&1\\1&1&1&0\end{pmatrix},\begin{pmatrix}0&1&1&1\\1&0&0&0\\1&0&1&1\\1&1&0&1\end{pmatrix},\begin{pmatrix}0&1&1&1\\1&0&0&0\\1&0&1&1\\1&1&1&0\end{pmatrix},\begin{pmatrix}0&1&1&1\\1&0&0&0\\1&1&0&1\\1&1&1&0\end{pmatrix},\begin{pmatrix}0&1&1&1\\1&0&0&1\\1&0&1&1\\1&1&0&1\end{pmatrix},\begin{pmatrix}0&1&1&1\\1&0&1&0\\1&0&1&1\\1&1&1&0\end{pmatrix},\begin{pmatrix}0&1&1&1\\1&0&1&0\\1&1&0&1\\1&1&1&0\end{pmatrix},\begin{pmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{pmatrix},$$

$$\begin{pmatrix}0&1&1&1\\1&1&0&0\\1&1&0&1\\1&1&1&0\end{pmatrix},\begin{pmatrix}1&0&0&0\\1&0&1&1\\1&1&0&0\\1&1&0&1\end{pmatrix},\begin{pmatrix}1&0&0&0\\1&0&1&1\\1&1&0&0\\1&1&1&0\end{pmatrix},\begin{pmatrix}1&0&0&0\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{pmatrix},\begin{pmatrix}1&0&1&1\\1&1&0&0\\1&1&0&1\\1&1&1&0\end{pmatrix}.$$

0, 0, 0, 58, 1629, 28924, 507052, 8211776, 133693904, . . .

$$-22912660668416(n-2)a(n)+4194304(4419089n-5784790)a(n+1)+458752(9499785n-97594504)a(n+2)-8192(890967049n-4679365255)a(n+3)+1024(1488027923n-5601351692)a(n+4)+6528(44221759n-387235809)a(n+5)-32(3992176883n-26858644798)a(n+6)+8(1107194741n-6399743425)a(n+7)+2(610453317n-4593544888)a(n+8)-(182139823n-1273140745)a(n+9)+(6061186n-41678719)a(n+10)\overset{?}{=}0$$

$$
\begin{aligned}
a(n) \overset{?}{=} {} & \tfrac{1}{3}2^{2n-11}(6n^2-219n+820)-\tfrac{1}{9}2^{n-5}(3n+32) \\
& -\tfrac{113}{3}(-1)^n 2^{3n-14}-\tfrac{1}{3}(-1)^n 2^{2n-11}(13n-164) \\
& +\tfrac{1}{9}2^{3n-14}(288n-3473)+2^{4n-9}
\end{aligned}
$$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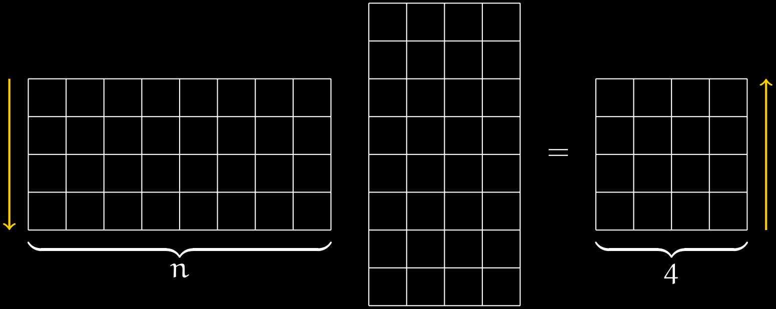
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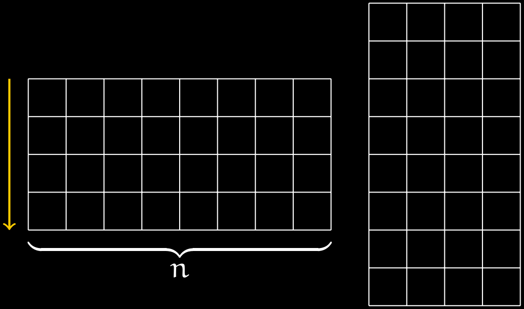
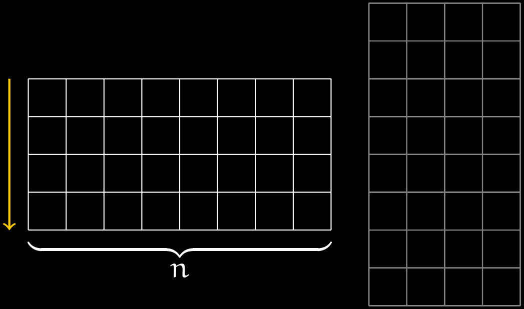
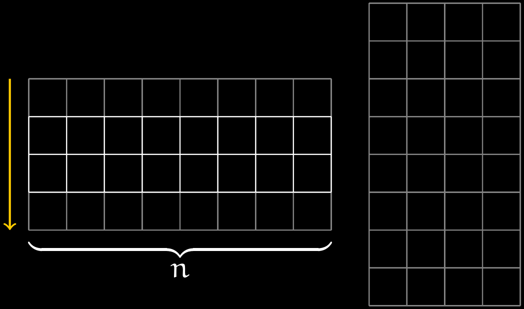


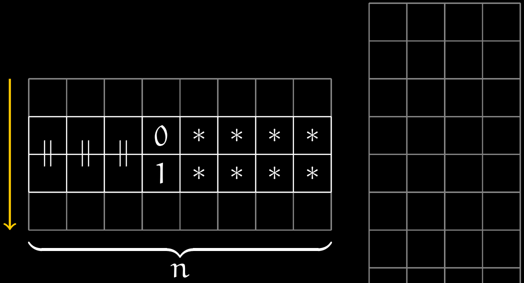
$= \; A \in \mathbb{Z}_2^{4\times 4}$ fixed



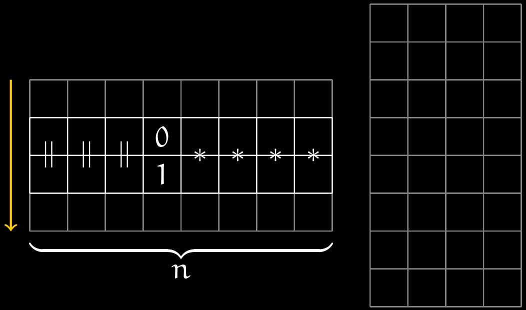
$= A \in \mathbb{Z}_2^{4\times 4}$ fixed



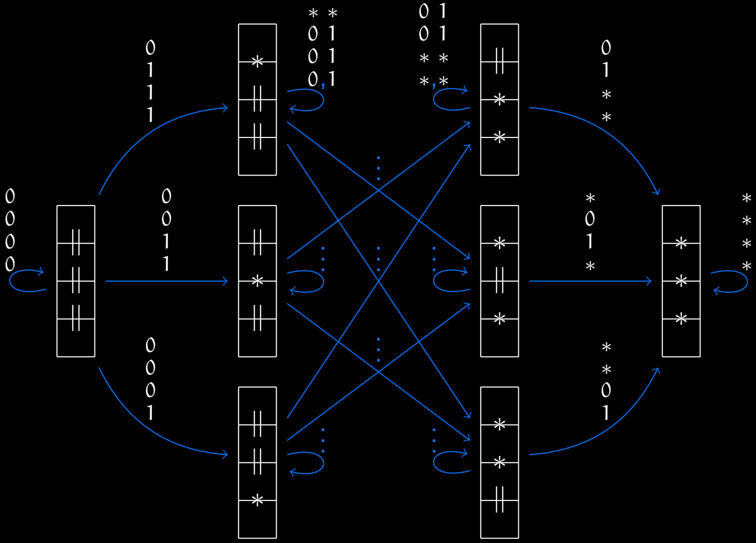
$= A \in \mathbb{Z}_2^{4\times 4}$ fixed



$= A \in \mathbb{Z}_2^{4\times 4}$ fixed



$= A \in \mathbb{Z}_2^{4\times 4}$ fixed


0 0 0 0
0 1 1 1
0 0 1 1
0 0 0 1
* * 0 1 0 1 0,1
0 1 0 1 * * *,*
0 1 * *
* 0 1 *
* * 0 1
* * * *

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 8 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 & 8 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 4 & 4 & 4 & 16 \end{pmatrix}$$

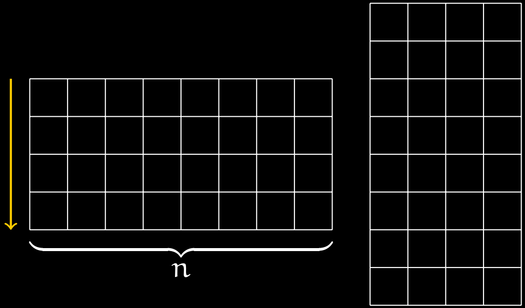
$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 8 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 & 8 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 4 & 4 & 4 & 16 \end{pmatrix}^n$$

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 8 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 & 8 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 4 & 4 & 4 & 16 \end{pmatrix}^n \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

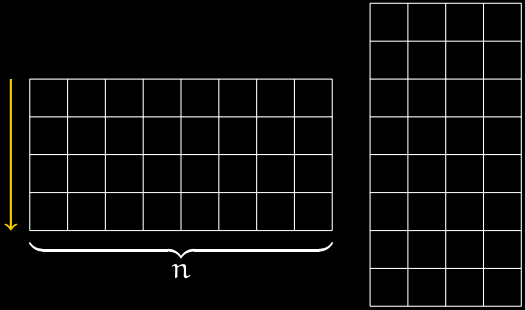
$$(0,0,0,0,0,0,0,1)\begin{pmatrix}2&0&0&0&0&0&0&0\\1&4&0&0&0&0&0&0\\1&0&4&0&0&0&0&0\\1&0&0&4&0&0&0&0\\0&0&2&2&8&0&0&0\\0&2&0&2&0&8&0&0\\0&2&2&0&0&0&8&0\\0&0&0&0&4&4&4&16\end{pmatrix}^n\begin{pmatrix}1\\1\\1\\1\\0\\0\\0\\0\end{pmatrix}$$

$$(0,0,0,0,0,0,0,1)\begin{pmatrix}2&0&0&0&0&0&0&0\\1&4&0&0&0&0&0&0\\1&0&4&0&0&0&0&0\\1&0&0&4&0&0&0&0\\0&0&2&2&8&0&0&0\\0&2&0&2&0&8&0&0\\0&2&2&0&0&0&8&0\\0&0&0&0&4&4&4&16\end{pmatrix}^{n}\begin{pmatrix}1\\1\\1\\1\\0\\0\\0\\0\end{pmatrix}$$

$=$ Number of $4 \times n$ binary matrices M with rows in strictly increasing order

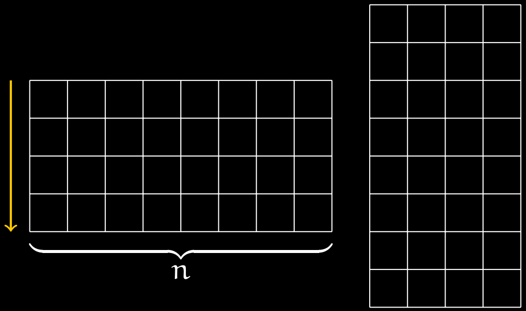


$= A \in \mathbb{Z}_2^{4\times 4}$ fixed



$= A \in \mathbb{Z}_2^{4\times 4}$ fixed

What is the effect on $a_{i,j}$ when we extend M by another column?



$= A \in \mathbb{Z}_2^{4\times 4}$ fixed

What is the effect on $a_{i,j}$ when we extend M by another column?

It will flip if and only if the new column has a 1 in rows i and j.

Refine the counting by introducing marker variables $x_{i,j}$ that keep track of the number of columns in M that have a 1 in rows i and j.

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Examples:

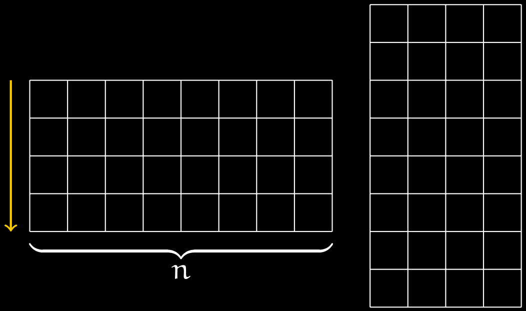
$$\begin{matrix}0\\0\\0\\0\end{matrix} \to 1 \quad \begin{matrix}0\\0\\1\\0\end{matrix} \to x_{3,3} \quad \begin{matrix}1\\0\\0\\1\end{matrix} \to x_{1,1}x_{1,4}x_{4,4} \quad \begin{matrix}0\\1\\1\\1\end{matrix} \to x_{2,2}x_{2,3}x_{2,4}x_{3,3}x_{3,4}x_{4,4}$$

$$\begin{matrix}1\\1\\1\\1\end{matrix} \to x_{1,1}x_{1,2}x_{1,3}x_{1,4}x_{2,2}x_{2,3}x_{2,4}x_{3,3}x_{3,4}x_{4,4}$$

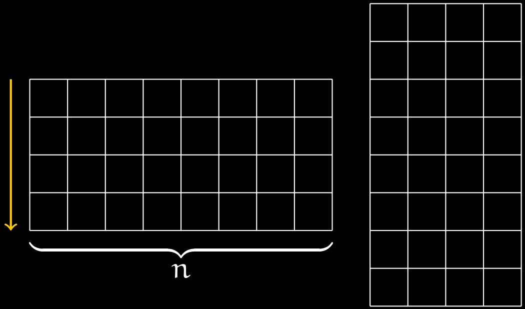
$$(0,0,0,0,0,0,0,1)\left(\boxed{\begin{array}{c}\text{messy lower triangular}\\ \text{matrix with}\\ \text{polynomial entries}\end{array}}\right)^{n}\begin{pmatrix}1\\1\\1\\1\\0\\0\\0\\0\end{pmatrix}$$

$$(0,0,0,0,0,0,0,1)\left(\boxed{\begin{array}{c}\text{messy lower triangular}\\ \text{matrix with}\\ \text{polynomial entries}\end{array}}\right)^{n}\begin{pmatrix}1\\1\\1\\1\\0\\0\\0\\0\end{pmatrix}$$

$=$ a polynomial p such that the coefficient of $x_{1,1}^{e_{1,1}} \cdots x_{4,4}^{e_{4,4}}$ is the number of $4 \times n$ binary matrices M with rows in strictly increasing order that have $e_{i,j}$ columns with a 1 in rows i and j, for all i, j.

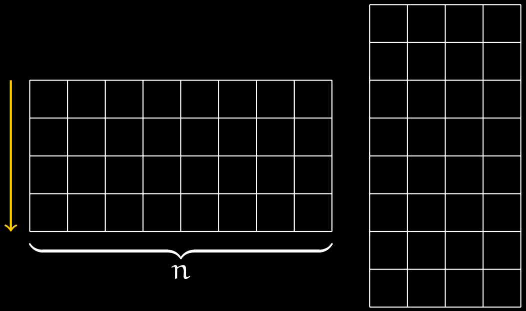


$= A \in \mathbb{Z}_2^{4\times 4}$ fixed



$= A \in \mathbb{Z}_2^{4\times 4}$ fixed

Matrices M yielding $a_{i,j} = 1$ are counted by monomials in which the exponent of $x_{i,j}$ is odd.



$= A \in \mathbb{Z}_2^{4\times 4}$ fixed

Matrices M yielding $a_{i,j} = 1$ are counted by monomials in which the exponent of $x_{i,j}$ is odd.

Matrices M yielding $a_{i,j} = 0$ are counted by monomials in which the exponent of $x_{i,j}$ is even.

The complete generating function f is rational in t and all the $x_{i,j}$.

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Let's say we fix $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \in \mathbb{Z}_2^{4\times 4}$.

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Let's say we fix $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \in \mathbb{Z}_2^{4\times 4}$.

Then the $4 \times n$ binary matrices M with rows in strictly increasing order and $MM^\top = A \pmod 2$ is counted by

$$\left([x_{1,1}^{\text{odd}} x_{1,2}^{\text{odd}} x_{1,3}^{\text{even}} x_{1,4}^{\text{even}} x_{2,2}^{\text{even}} x_{2,3}^{\text{odd}} x_{2,4}^{\text{odd}} x_{3,3}^{\text{odd}} x_{3,4}^{\text{even}} x_{4,4}^{\text{odd}}] f\right)\Big|_{x_{i,j}=1}.$$

The complete generating function f is rational in t and all the $x_{i,j}$.

Let's say we fix $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \in \mathbb{Z}_2^{4\times 4}$.

Then the $4 \times n$ binary matrices M with rows in strictly increasing order and $MM^\top = A$ (mod 2) is counted by

$$\left([x_{1,1}^{\text{odd}} x_{1,2}^{\text{odd}} x_{1,3}^{\text{even}} x_{1,4}^{\text{even}} x_{2,2}^{\text{even}} x_{2,3}^{\text{odd}} x_{2,4}^{\text{odd}} x_{3,3}^{\text{odd}} x_{3,4}^{\text{even}} x_{4,4}^{\text{odd}}] f\right)\Big|_{x_{i,j}=1}.$$

This is a rational function in t.

Theorem.

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Theorem.

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Then $(a_n)_{n=0}^{\infty}$ is C-finite.

Theorem.

- Let $k \in \mathbb{N}$.
- Let $\mathcal{A} \subseteq \mathbb{Z}_2^{k \times k}$.
- Let a_n be the number of matrices $M \in \mathbb{Z}_2^{k \times n}$ with rows in strictly increasing order and $MM^\top \in \mathcal{A}$.

Then $(a_n)_{n=0}^{\infty}$ is C-finite.

In particular, the sequence A181280 is C-finite. ■

Entry	First terms	N_{oeis}	N_{linalg}	N_{LLL}	r	d	
A177317	1, 2, 48, 2288, 135040	29	60	22	3	14	✓
A199250	1, 1, 14, 21, 424, 571	56	98	56	8	18	✓
A250556	8, 60, 302, 1516, 7126	47	58	47	9	8	✓
A265234	1, 43, 2592, 184740	31	56	27	6	6	✓
A172572	90, 67950, 90291600	33	44	17	3	9	✓
A172671	90, 202410, 747558000	33	75	25	4	13	✓
A188818	2, 9, 48, 256, 1360	32	55	26	5	10	✓
A306322	1, 0, 0, 25, 386, 4657	41	63	30	4	14	✓
A195806	16, 105, 496, 1759, 5052	32	41	30	4	10	✓
A216940	260, 27768, 1664244	37	44	29	1	23	✓
A194478	0, 0, 0, 1, 337, 8733	32	35	19	2	12	✓
A215570	1, 35, 18720, 19369350	48	68	27	3	15	?
A339987	1, 4, 90, 8400, 1426950	40	70	24	5	10	✓
A269021	1, 2, 23, 588, 24553	42	108	28	4	21	?
A181198	1, 1, 8, 169, 6392	27	33	14	2	9	?
A181199	1, 1, 16, 985, 141696	26	103	34	3	24	?
A181280	0, 0, 0, 58, 1629, 28924	27	32	26	10	1	✓
A253217	0, 0, 1, 19, 268, 3568	37	53	27	5	9	✓
A098926	0, 2, 12, 90, 556, 5242	34	55	26	8	7	?
A164735	0, 0, 0, 0, 0, 0, 0, 1, 0, 4	70	80	66	15	4	?

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A195806	16, 105, 496, 1759, 5052	32	41	30	4	10	✓
A216940	260, 27768, 1664244	37	44	29	1	23	✓
A194478	0, 0, 0, 1, 337, 8733	32	35	19	2	12	✓
A215570	1, 35, 18720, 19369350	48	68	27	3	15	?
A339987	1, 4, 90, 8400, 1426950	40	70	24	5	10	✓
A269021	1, 2, 23, 588, 24553	42	108	28	4	21	?
A181198	1, 1, 8, 169, 6392	27	33	14	2	9	?
A181199	1, 1, 16, 985, 141696	26	103	34	3	24	?
A181280	0, 0, 0, 58, 1629, 28924	27	32	26	10	1	✓
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3 4 2 7 9 1 5 6 8

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3	4	2	7	9	1	5	6	8
		1	3			2		4

3 4 2 7 9 1 5 6 8

1 3 2 4

How many permutations contain the pattern 1324?

3 4 2 7 9 1 5 6 8

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Let a_n be the number of such permutations in S_n

3 4 2 7 9 1 5 6 8

1 3 2 4

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Let a_n be the number of such permutations in S_n

Famous open problem: is a_n D-finite?

3 4 2 7 9 1 5 6 8

3	4	2	7	9	1	5	6	8
1	2		3					4

3	4	2	7	9	1	5	6	8
1	2		3					4

How many permutations contain the pattern 1234?

3	4	2	7	9	1	5	6	8
1	2		3					4

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Let a_n be the number of such permutations in S_n

3 4 2 7 9 1 5 6 8

1 2 3 4

How many permutations contain the pattern 1234?

Let a_n be the number of such permutations in S_n

Well-known fact: a_n is D-finite.

3	4	2	7	9	1	5	6	8
1	2		3					4

How many permutations contain the pattern $12\ldots k$?

Let a_n be the number of such permutations in S_n

Well-known fact: a_n is D-finite *for every fixed* k*.*

A269021: $a_n =$ number of permutations of length $2n$, containing the pattern $12\ldots n$.

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Conjecture: This sequence is D-finite.

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Conjecture: This sequence is D-finite.

$$-64(n+1)^2(n+2)^2(n+3)(2n+1)^2(2n+3)^2(2n+5)^2(64n^{10}+1968n^9+26156n^8+198469n^7+952323n^6+3012795n^5+6333869n^4+8663374n^3+7264534n^2+3266000n+549760)a_n+16(n+2)^2(n+3)(2n+3)^2(2n+5)^2(64n^{13}+2672n^{12}+49788n^{11}+545913n^{10}+3917758n^9+19359535n^8+67385886n^7+165789363n^6+284054698n^5+325846005n^4+229526554n^3+78563984n^2-487964n-5543040)a_{n+1}-4(n+3)(2n+5)^2(512n^{15}+21568n^{14}+419248n^{13}+4969164n^{12}+39928763n^{11}+228837227n^{10}+959068672n^9+2966908118n^8+6753094929n^7+11118771121n^6+12741784568n^5+9313604242n^4+3271711596n^3-562569136n^2-946158512n-250467360)a_{n+2}+2(512n^{16}+26752n^{15}+624800n^{14}+8677944n^{13}+80260596n^{12}+523718876n^{11}+2488583381n^{10}+8747566435n^9+22820793074n^8+43766004538n^7+60004107039n^6+55047935941n^5+27672902302n^4-778719870n^3-10812498240n^2-6360099840n-1300242000)a_{n+3}-3(n+4)(3n+8)(3n+10)(64n^{10}+1328n^9+11324n^8+52389n^7+143536n^6+233810n^5+204716n^4+48699n^3-68928n^2-61278n-15900)a_{n+4}=0$$


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