Experimenting with Permutation Wordle

Experimental Math Seminar 5/1

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Permutation Wordle

1. I'm thinking of $\pi \in \mathcal{S}_n$
2. Your goal: guess π in as few guesses as possible
3. After each guess, I will tell you which indices (if any) are correct
4. The game ends when you have guessed my permutation

Example

$$\gamma_1 = \quad 1\mathbf{2}345 \qquad \begin{cases} \mathcal{I}_1 = \{1,3,4,5\} \\ \mathcal{J}_1 = \{2\} \end{cases}$$

$$\gamma_2 = \quad 5\mathbf{2}1\mathbf{3}4 \qquad \begin{cases} \mathcal{I}_2 = \{1,3,5\} \\ \mathcal{J}_2 = \{2,4\} \end{cases}$$

$$\gamma_3 = \quad \mathbf{42531} \qquad \begin{cases} \mathcal{I}_3 = \emptyset \\ \mathcal{J}_3 = \{1,2,3,4,5\} \end{cases}$$

Cyclic Shift Conjecture

Def. In *cyclic shift* (abbreviated *CS*):

- $\gamma_{k+1}[j_a] = \gamma_k[j_a]$ for $j_a \in \{j_1, \ldots, j_{|\mathcal{J}_k|}\} = \mathcal{J}_k$
- $\gamma_{k+1}[i_b] = \gamma_k[i_{b-1}]$ for $i_b \in \{i_1, \ldots, i_{|\mathcal{I}_k|}\} = \mathcal{I}_k$

Example.

$$\mathbf{1}\ \ 2\ \ 3\ \ 4\ \ 5 \quad \Longrightarrow \quad \mathbf{1}\ \ 5\ \ \mathbf{2}\ \ 3\ \ 4 \quad \Longrightarrow \quad \mathbf{1}\ \ \mathbf{4}\ \ \mathbf{2}\ \ \mathbf{5}\ \ \mathbf{3}$$

Conjecture. *CS* attains the maximal probability of success in r guesses

Alt: *CS* has the smallest average number of guesses for $\pi \in \mathcal{S}_n$

Strategy Formalization

- $S := [S_1, S_2, \ldots, S_n]$ where $S_i \in \mathcal{S}_i$ for all $i \in [n]$
- Generate γ_{k+1} from γ_k using $S[|\mathcal{I}_k|] = S_{|\mathcal{I}_k|}$

Example:

- $\gamma_2 = 5\mathbf{2}1\mathbf{3}4$, $\mathcal{I}_2 = \{1, 3, 5\} \implies$ use $S[3] \in \mathcal{S}_3$
- If $S[3] = 231$, then $514 \mapsto 451$
- $\gamma_3 = 42531$

Alternative:

- If $S[3] = 132$, then $514 \mapsto 541$
- $\gamma_3 = 52431$

Cyclic Shift & Other Strategies

Cyclic Shifts

$CS := [[1], [2,1], [2,3,1], \ldots, [2,3,\ldots,n,1]]$
$CS' := [[1], [2,1], [3,1,2], \ldots, [n,1,2,\ldots,n-1]]$

Cyclic Strategies:
$S := [C_1, C_2, C_3, \ldots, C_n]$ where C_i cyclic perm of $[i]$
$\implies S := [[1], [2,1], C_3, \ldots, C_n]$
$\implies C_3 = [2,3,1]$ or $[3,1,2]$

Motivation:

- No fixed points
- Smaller scope
- Easier to generate/analyze, recursive patterns

Later: $S := [D_1, D_2, D_3, \ldots, D_n]$ where $D_i \in Der(i)$

Generating Functions

$f_S(x) = \sum_{m \geq 1} a_m x^m$ where $a_m = \#\pi \in \mathcal{S}_n$ guessed in m guesses by strategy S

Example: $S = [[1], [2,1], [2,3,1]]$ has $f_S(x) = x + 4x^2 + x^3$

Remarks:

- For any S, coeff($f_S(x)$,1) $= 1$ (trivial perm.)
- Maximum degree not fixed (bounded?)
- Coeffs of $f_{CS}(x)$ are Eulerian numbers! (Kutin-Smithline)

Excedances & Eulerian Numbers

Def. $\text{Exc}(\pi) = \{i \mid \pi(i) > i\}$

Examples:

$\pi = 132 \implies \text{Exc}(\pi) = \{2\}$
$\pi = 231 \implies \text{Exc}(\pi) = \{1, 2\}$

Claim (Kutin-Smithline)

CS guesses π in $1 + |\text{Exc}(\pi)|$ guesses

★ $A(n,k) = \#\pi \in \mathcal{S}_n$ s.t. $|\text{Exc}(\pi)| = k$

Proof.

Omitted for time (see Kutin-Smithline Section 3) □

Inductive Approach

CSInd $:= [[1], [2, 1], [2, 3, 1], \dots, [2, 3, \dots, n-1, 1], d]$
$\hookrightarrow d \in Der(n)$

Test Performance:

- $\pi \in Der(n)$
- Stop when $\mathcal{J} \neq \emptyset$
- **Q:** How long does it take before something is correct? How many correct indices? On average?
- **A:** Surprisingly, CSInd(d) for all d is the *same*...

Inductive Strategy Example

Derangement	$S[n] := [2,3,4,1]$ $\gamma_2 = [4,1,2,3]$	$S[n] := [2,1,4,3]$ $\gamma_2 = [2,1,4,3]$
$[2,1,4,3]$	$\{2,4\}$	$\{1,2,3,4\}$
$[2,3,4,1]$	$\emptyset$	$\{1,3\}$
$[2,4,1,3]$	$\{4\}$	$\{1,4\}$
$[3,1,4,2]$	$\{2\}$	$\{2,3\}$
$[3,4,1,2]$	$\emptyset$	$\emptyset$
$[3,4,2,1]$	$\{3\}$	$\emptyset$
$[4,1,2,3]$	$\{1,2,3,4\}$	$\{2,4\}$
$[4,3,1,2]$	$\{1\}$	$\emptyset$
$[4,3,2,1]$	$\{1,3\}$	$\emptyset$
Total	12	12
Average	$4/3$	$4/3$

Claim. Average over $\pi \in Der(n)$ of $|\mathcal{J}_2| = \frac{n}{n-1}$

One (Mandatory) Proof

Claim. For any $d \in Der(n)$, $\sum_{\pi \in D_n} |\{i \mid \pi(i) = \delta(i)\}| = \frac{n}{n-1} \cdot D_n$
★ Note: $\delta = d^{-1} = (S[n])^{-1} = \gamma_2$

Proof.

Equivalent to counting matching entries.
Recall $D_n = (n-1)(D_{n-1} + D_{n-2})$.

$$\begin{aligned}
\sum_{\pi \in D_n} |\{i \mid \pi(i) = \delta(i)\}| &= \sum_{i=1}^{n} |\{\pi \in D_n \mid \pi(i) = \delta(i)\}| \\
&= \sum_{i=1}^{n} (D_{n-1} + D_{n-2}) \\
&= n \cdot (D_{n-1} + D_{n-2}) \\
&= n \cdot \left(\frac{D_n}{n-1} \right)
\end{aligned}$$

□

What's Next?

- Relationships between components?
- Duplicated information $\implies$ BAD!
  Example: $S[4] = [2, 1, 4, 3]$, $\pi = 3421$

$$\begin{array}{ll}
\gamma_1 = [1,2,3,4] & \mathcal{J}_1 = \emptyset \\
\gamma_2 = [2,1,4,3] & \mathcal{J}_2 = \emptyset \\
\gamma_3 = [1,2,3,4] & \mathcal{J}_3 = \emptyset \\
\gamma_4 = [2,1,4,3] & \mathcal{J}_4 = \emptyset \\
\vdots & \vdots
\end{array}$$

Generating Function Observations:

- All strategies have the same coeffs for x (duh) and x^2
- $\text{coeff}(f_{CS}(x), 3)$ is maximum over *all* strategies

Thanks & Questions