Young Tableau Reconstruction Via Minors

Will Erickson, Daniel Herden*, Jonathan Meddaugh, Mark Sepanski, Cordell Hammon, Jasmin Mohn, Indalecio Ruiz-Bolanos

Baylor University

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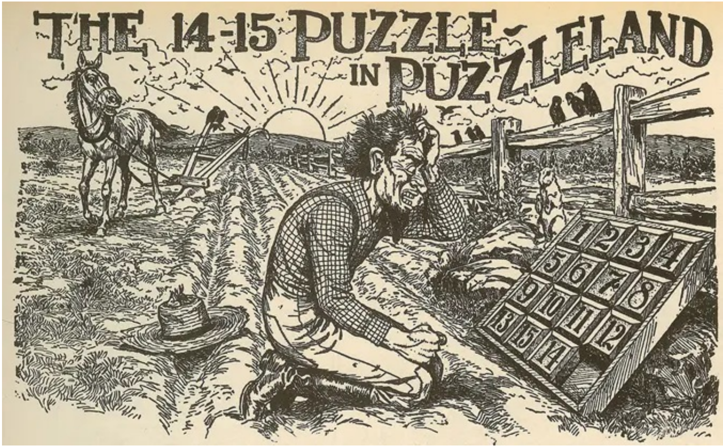
Old Time Games

The *15 Puzzle* (also called Gem Puzzle, Boss Puzzle, Game of Fifteen, Mystic Square), *Taquin* (Teasing) in French, is a game dating to 1874.

The 15 Puzzle

The 14-15 Puzzle (Sam Loyd, 1880s)



Sam Loyd: Solve the puzzle and win \$1000!!!

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Optimal solutions can vary from 0 to 80 single-tile moves.
There are 17 configurations that require 80 moves!

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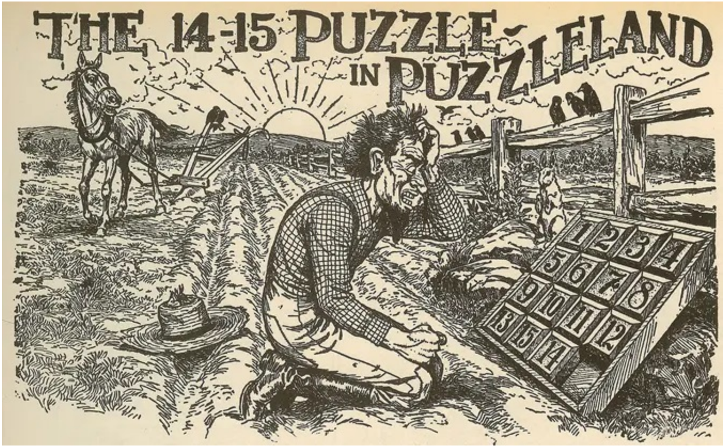

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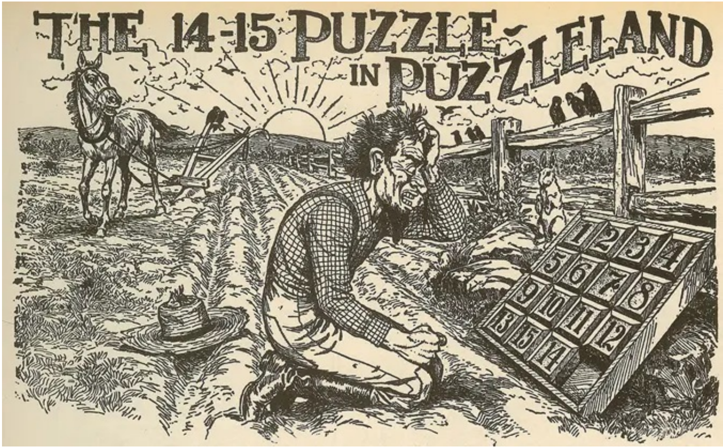

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Some Motivating Problems

Jeu de Taquin

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1. Reconstruction problem of partitions of n from k-minors

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Origin of our Young tableau reconstruction problem:

1. Reconstruction problem of partitions of n from k-minors
2. Recovery of characters of S_n by restriction to subgroups

Definition of Partition

For $n \in \mathbb{Z}_{>0}$, a partition of n is a weakly decreasing finite sequence

$$\lambda = (\lambda_1, \ldots, \lambda_m)$$

of positive integers such that $\sum_i \lambda_i = n$. We call n the *size* of λ.

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For example,

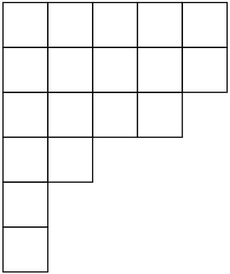
$$\lambda = (5, 5, 4, 2, 1, 1)$$

is a partition of size 18.

Definition of Young Diagram

The Young diagram of shape λ is a left-aligned array of cells with λ_i boxes in the ith row, counting from the top.

As an example, the Young diagram of shape $(5, 5, 4, 2, 1, 1)$ is:



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We write YT(n) for the set of standard Young tableaux of size n.

1	2	3	4	5
6	7	8	9	18
10	11	12	17	
13	16			
14				
15				

A standard Young tableau of shape $(5, 5, 4, 2, 1, 1)$

Definition of Outer Corner

Given a tableau $T \in \mathrm{YT}(n)$, an outer corner (OC for short) is a cell of T which is both the right end of a row and the bottom end of a column of T.

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The OCs are 15, 16, 17, and 18

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Cleanest example is

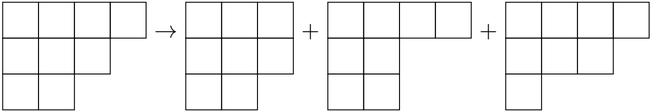
$$\text{irreducible representations of } S_n \leftrightarrow \text{partitions of } n.$$

Reconstruction from Restriction?

The branching law for restricting from

$$S_n \to S_{n-1}$$

is given by all possible ways of removing an OC.



Restriction from $S_9 \to S_8$

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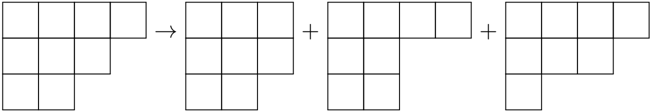

Restriction from $S_9 \to S_8$

Question: When does the restriction determine the original representation?

Deleting a Cell

There is a natural way of deleting cells from a tableau, T, using the process known as jeu de taquin, introduced by Schützenberger in 1977. If m is the deleted cell, write $T - m$ for the result.

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- if there exists a cell either directly to the right or directly below the empty space, slide the cell with the smaller entry into the position of the empty space.

This terminates when there are no cells directly to the right or below the current empty space. Finally:

- subtract 1 from each entry larger than m.

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$T - 8$ via jeu de taquin

The Reconstructibility Question

For $T \in \mathrm{YT}(n)$, a k-minor of T is a tableau formed by iteratively deleting k cells from T via jeu de taquin. We write $\mathrm{M}_k(T)$ for the set of all k-minors of T and $\mathrm{mM}_k(T)$ for the multiset of all k-minors of T.

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- Cain, Lehtonen (2022): for $k = 1$, every tableau of size n can be reconstructed from its set of 1-minors when $n \geq 5$.

Our Main Results

Theorem

Let $n \in \mathbb{Z}_{>0}$ *and* $T \in \mathsf{YT}(n)$*. Then* $\mathsf{M}_2(T)$ *determines* T *when* $n \geq 8$*.*

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Let $T \in \mathrm{YT}(n)$*. The multiset* $\mathrm{mM}_k(T)$ *determines* T *when*

$$n > \frac{k^3 + 2k^2}{\ln 2} + \frac{k^2}{2} + 2k - 1 + \frac{\ln 2}{12}(k+2)$$

as a cubic lower bound.

Removal of Boxes

Let $T \in \mathrm{YT}(n)$. Write

$$\mathrm{R}_n\, T$$

for the tableau in $\mathrm{YT}(n-1)$ obtained by removing the cell with label n from T.

More generally for $d \in [1, n]$, write $\mathrm{R}_{[d,n]}\, T$ for the tableau

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Lemma

Let $n, k \in \mathbb{Z}_{>0}$, let $d \in [k+1, n]$, and let $T \in \mathrm{YT}(n)$. Then

$$\mathrm{M}_k(\mathrm{R}_{[d,n]}\, T) = \mathrm{R}_{[d-k,n-k]}\, \mathrm{M}_k(T).$$

Many Entries Determined by Removal of Boxes

Lemma

Let $T \in \mathrm{YT}(n)$*. If*

$$n \geq k^2 + 2k + 1,$$

then the location in T *of each entry in* $[(k+1)^2, n]$ *is determined by* $\mathrm{M}_k(T)$*.*

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Idea: recall the shape of T is determined by the shapes in $\mathrm{M}_k(T)$ when $n \geq k^2 + 2k$. When $n - 1 \geq k^2 + 2k$, the shape of $\mathrm{R}_n\, T$ is determined by $\mathrm{M}_k(\mathrm{R}_n\, T)$ which, in turn, is the same as $\mathrm{R}_{n-k}\, \mathrm{M}_k(T)$.
Since taking the complement of the shape of $\mathrm{R}_n\, T$ in the shape of T gives us the location of n in T, it follows that $\mathrm{M}_k(T)$ determines the location of n when $n \geq k^2 + 2k + 1 \ldots$

General Multiset Reconstruction Lower Bound

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Idea: it remains to find the location of $1, 2, \ldots, k^2 + 2k$. For m in this range, its location may be identified as the most frequent location of m in the multiset of k-minors when there are enough larger elements, i.e.,

$$\binom{n-m}{k} > \frac{1}{2}\binom{n}{k}.$$

Conjectural Multiset Bound

Conjecture

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Recall

Theorem

Let $n \in \mathbb{Z}_{>0}$ *and* $T \in \mathrm{YT}(n)$. *Then* $\mathrm{M}_2(T)$ *determines* T *when* $n \geq 8$.

Improved Location of n

We know the location of n when $n \geq k^2 + 2k + 1$. In fact, we need a bit more.

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Lemma

Let $k \geq 2$ and $T \in \mathrm{YT}(n)$. Then $\mathrm{M}_k(T)$ determines the location of n when

$$n \geq k^2 + 2k.$$

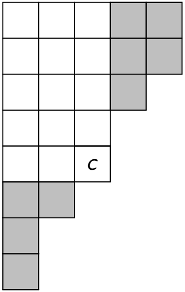
This result proceeds in stages.

At Most One Small OC

Lemma

Let $T \in \mathrm{YT}(n)$ with OCs $c_1, \ldots, c_\ell$. Suppose $|\mathrm{Out}(c_j)| \leq k$ for at most one OC. Then the location of n is recoverable from $\mathrm{M}_k(T)$.

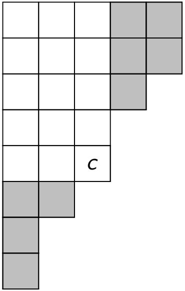
If c is an OC of T, we define its outer area, denoted by $\mathrm{Out}(c)$, to be the collection of all cells of T that are to the right of c or below c.



$\mathrm{Out}(c)$

At Most One Small OC

Idea: if c contains entry n, there exist tableaux in $\mathrm{M}_k(T)$ where c survives with entry $n-k$.



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Only One With Multiple Small OCs

If at least two OCs c_j have $|\operatorname{Out}(c_j)| \leq k$, it turns out that the configuration for T is very limited, at least when $n \geq k^2 + 2k$, which is the lower bound for recovering the shape of T from the shapes of $\mathrm{M}_k(T)$.

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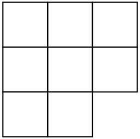
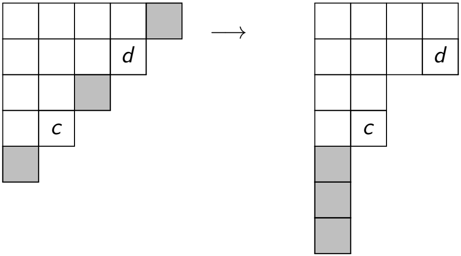
Lemma

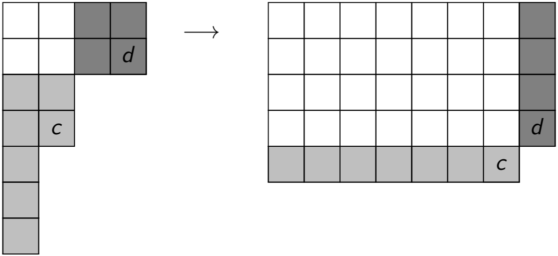
Let $T \in \mathrm{YT}(n)$ with OCs $c_1, \ldots, c_\ell$. Suppose $|\mathrm{Out}(c_j)| \leq k$ for at least two OCs. Then $|T| \leq k^2 + 2k$. Equality holds if and only if T has shape $((k+1)^k, k)$, i.e., the shape of a $(k+1) \times (k+1)$ square with the lower-right cell removed.



Largest shape with multiple OCs such that $|\mathrm{Out}(c_j)| \leq k$, where $k = 3$

Algorithm for Multiple Small OCs

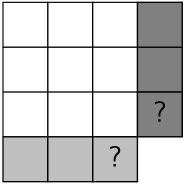


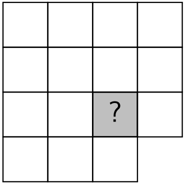


Determination of n for the Bad Shape

Theorem

Let $k \geq 2$, $n = (k+1)^2 - 1$, and $T \in \mathrm{YT}(n)$ with shape $((k+1)^k, k)$. Then $\mathrm{M}_k(T)$ determines the location of n.





Down to 8

Recall that we want to show:

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Let $T \in \mathrm{YT}(n)$. *Then* $\mathrm{M}_2(T)$ *determines* T *when* $n \geq 8$.

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Idea: we know that $\mathrm{M}_2(T)$ determines the shape of T and location of $[9, n]$. Thus, only the location of $[1, 8]$ remains to be determined. As

$$\mathrm{M}_2(R_{[9,n]} T) = R_{[7,n-2]} \mathrm{M}_2(T),$$

it suffices to show the following:

Lemma

Let $T \in \mathrm{YT}(8)$*. Then* $\mathrm{M}_2(T)$ *determines* T*.*

Down to 6

We can significantly reduce the number of cases to check using a more refined determination of shape:

Lemma (Monks, 09)

Let $T \in \mathrm{YT}(n)$*. Then* $M_2(T)$ *determines the shape of* T *if* n *cannot be expressed as* $n = (a+1)b + c - 1$ *for* $a, b, c \in \mathbb{Z}_{>0}$ *satisfying*

$$a \leq c \leq 2 \quad \textit{and} \quad b + (c \mod a) \leq 2.$$

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$$a \leq c \leq 2 \quad \textit{and} \quad b + (c \mod a) \leq 2.$$

From this, we see the shape of T is recoverable from $\mathrm{M}_2(T)$ when $n = 6$. Therefore $M_2(R_{[7,8]}T)$ determines the shape of

$$T' = R_{[7,8]}T.$$

Since 7 and 8 are in the complement of T' and the location of 8 is known, the location of 7 in T is also determined.

6 and Below... and Its 6 Cases!

Thus it remains to show that $M_2(T)$ determines $T' = R_{[7,8]}T$. Recall that the 2-minors of T' are $M_2(T') = R_{[5,6]}M_2(T)$.

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Up to symmetry, there are 6 shapes of T' to consider:

1. (6)
2. $(5, 1)$
3. $(4, 1, 1)$
4. $(4, 2)$
5. $(3, 3)$
6. $(3, 2, 1)$

The first three can be done simultaneously, while the last three must be analyzed individually.

Conjectural General Bound

Conjecture

Let $k \geq 2$. *Let* $T \in \mathrm{YT}(n)$, *where* $n \geq k^2 + 2k$. *Then* $\mathrm{M}_k(T)$ *determines* T.

The End

Thank you!

Sam Loyd: Find the 5-pointed star!

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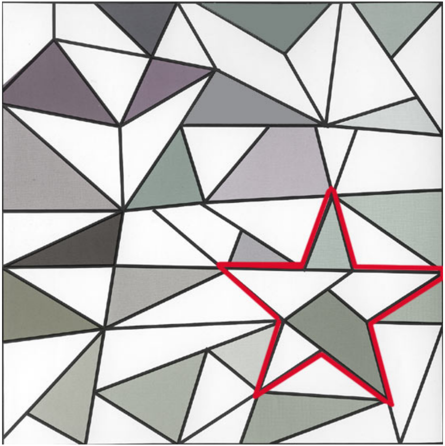

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