Bilateral rational Ramanujan series and their p-adic mates

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The identity

$$x + x^2 + x^3 + x^4 + \cdots = x(1-x)^{-1}, \quad |x| < 1,$$

has sense in the reals. Replacing x with p (a prime), we have

$$p + p^2 + p^3 + p^4 + \cdots = p(1-p)^{-1},$$

which has no sense in the reals, but has sense in the p-adics because

$$p + p^2 + p^3 + \cdots + p^k \equiv p(1-p)^{-1} \pmod{p^{k+1}}, \quad k \geq 1.$$

For $p = 2$ (2-adic), we have $0 \cdot 2^0 + 1 \cdot 2^1 + 1 \cdot 2^2 + 1 \cdot 2^3 + \cdots = -2$. Hence ...11111110 $= -2$, and $-2 + 2 = 0$. Indeed

$$...111111111110 + ...000000000010 = ...000000000000 \quad \text{2-adic.}$$

Let $f(n) = (n+1)^{-2}$. It is easy to prove that

$$x^2 \sum_{n=0}^{\infty} (f(n) - f(n+x)) = 2\zeta(3)x^3 - 3\zeta(4)x^4 + \cdots .$$

$$\text{Denote} \quad S(N) = N^2 \sum_{n=0}^{N-1} f(n).$$

Let $x = \nu p$. We conjecture the following p-adic identity

$$S(\nu p) = S(\nu) + 2\zeta_p(3)\nu^3 p^3 + 4\zeta_p(5)\nu^5 p^5 + \cdots ,$$

$$\zeta_p(k) \equiv \frac{B_{p-k}}{k} = \zeta(1+k-p) \pmod{p},$$

$$\zeta_p(k) \equiv \frac{B_{p^{n-1}(p-1)+1-k}}{k-1} \left(1 - \frac{p^{n-1}}{k-1}\right) \pmod{p^n}, \quad n \geq 2.$$

We can write the rational Ramanujan-like series as

$$\sum_{n=0}^{\infty} R(n) = \sum_{n=0}^{\infty} \left(\prod_{i=0}^{2m} \frac{(s_i)_n}{(1)_n} \right) \sum_{k=0}^{m} a_k n^k z_0^n = \frac{\sqrt{(-1)^m \chi}}{\pi^m},$$

where z_0 is a rational, $a_0, a_1, \dots, a_m$ are positive rationals, and χ the discriminant of a certain quadratic field (imaginary or real), which is an integer. Below, we show an example

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^7 \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^9 \, 2^{12n}} (43680n^4 + 20632n^3 + 4340n^2 + 466n + 21) = \frac{2^{11}}{\pi^4}.$$

conjectured by Jim Cullen, and recently proved by Kam Cheong Au, using the WZ method.

We define

$$f(x)=\left(\sum_{n=0}^{\infty}R(n)-\sum_{n=-\infty}^{\infty}R(n+x)\right)e^{-i\pi x}\prod_{s_k}\frac{\cos\pi x-\cos\pi s_k}{1-\cos\pi s_k}.$$

If s_k is in the Ramanujan series then $1-s_k$ also is. As the function $f(x)$ is periodic and holomorphic it admits a Fourier expansion. In addition $f(x)=\mathcal{O}(e^{(2m+1)\pi)|Im(x)|}$, and so it terminates at $k=m$:

$$f(x)=\frac{\sqrt{(-1)^m\chi}}{\pi^m}\sum_{k=1}^{m}\left(\alpha_k(\cos 2\pi kx-1)+\beta_k\sin 2\pi kx\right),$$

where α_k and β_k are the coefficients. We will denote the extended series to the right and to the left by

$$R_x(+)=\sum_{n=0}^{\infty}R(n+x),\quad R_x(-)=\sum_{n=1}^{\infty}R(-n+x).$$

As the bilateral identity holds for all values of x, we can use it to get approximations of the $m+1$ values of a_k, the m values of α_k and the m values of β_k. For that aim we construct a linear system of $3m+1$ equations. Unfortunately, as we cannot solve the system in an exact way due to the infinite series, we only get approximations. Later, we will see that the p-adic mate of the bilateral series comes in our help allowing to obtain the exact values. Unfortunately we could not prove the p-adic version, and so it is a conjecture up to now. The sum to the left is equal to

$$R_x(-) = x^{2m+1} \sum_{n=1}^{\infty} \left(\prod_{i=0}^{2m} \frac{(1)_{n-x}}{(s_i)_{n-x}} \right) \sum_{k=0}^{m} a_k (-n+x)^{k-2m-1} \, z_0^{-n+x}.$$

If the values of x that we take are very small then the sum to the left is very small as well, and we can ignore it.

Developping the sum to the left side, we have

$$\sum_{n=0}^{\infty} R(n) - \sum_{n=0}^{\infty} R(n+x) = \frac{\sqrt{(-1)^m \chi}}{\Gamma(\frac{1}{2})^{2m}}$$

$$\times\, e^{i\pi x} \prod_{s_k} \frac{1-\cos \pi s_k}{\cos \pi x - \cos \pi s_k} \sum_{k=1}^{m} (\alpha_k(\cos 2\pi kx - 1) + \beta_k \sin 2\pi kx)$$

$$+ (A + Bx + Cx^2 + \cdots)x^{2m+1}, \quad |x| < 1.$$

We conjecture that A is of the form $A = rL_\chi(m+1)$, where r is a rational number.

Denote

$$S(N)=N^{-m}z_0^{-\nu}\left(\prod_{i=0}^{2m}\frac{(1)_\nu}{(s_i)_\nu}\right)\sum_{n=0}^{N-1}\left(\prod_{i=0}^{2m}\frac{(s_i)_n}{(1)_n}\right)\sum_{k=0}^{m}a_k n^k z_0^n,$$

We have

$$\sum_{n=0}^{\infty}R(n)=\frac{\sqrt{(-1)^m\chi}}{\pi^m}=\sqrt{\frac{(-1)^m\chi}{\Gamma\left(\frac{1}{2}\right)^{4m}}},\quad \text{(Ramanujan series).}$$

If $R_x(-)$ is the extended series to the left, we have

$$x^{-m}R_x(-)=rL_\chi(m+1)x^{m+1}+Bx^{m+2}+\cdots.$$

p-adic mate theorem (G.): For $\nu=1,2,3,\cdots:$, we have

$$S(\nu p)=\left(\frac{\chi}{p}\right)S(\nu)+rL_{\chi,p}(m+1)\nu^{m+1}p^{m+1}+B_p\nu^{m+2}p^{m+2}+\cdots.$$

Denote

$$S(X)=X^{-m}z_0^{-x}\left(\prod_{i=0}^{2m}\frac{(1)_x}{(s_i)_x}\right)\left(\sum_{n=0}^{\infty}R(n)-\sum_{n=-\infty}^{\infty}R(n+X)\right)_p.$$

The function $g(x)=S(xp)/S(x)$ is periodic of period $x=1$ and holomorphic. Hence, it admits a p-adic Fourier expansion

$$\frac{S(xp)}{S(x)}=\left(\frac{\chi}{p}\right)+\sum_{k=1}^{m}(\alpha_k(\cos_p 2\pi kx-1)+\beta_k\sin_p 2\pi kx).$$

Replacing x with $\nu=1,2,3,\dots,$ we see that

$$S(\nu p)=\left(\frac{\chi}{p}\right)S(\nu)+rL_{\chi,p}(m+1)\nu^{m+1}p^{m+1}+B_p\nu^{m+2}p^{m+2}+\cdots.$$

The p-adic mate theorem (G.) implies the following supercongruences for $\nu = 1, 2, 3, \dots$ conjectured in 2008 for $\nu = 1$ by Wadim Zudilin:

$$S(\nu p) = S(\nu) \left(\frac{\chi}{p} \right) \pmod{p^{m+1}}, \quad \nu = 1, 2, 3, \dots,$$

and also the following supercongruences for $\nu = 1, 2, 3, \dots$, conjectured for $\nu = 1$ in 2018 by Yue Zhao:

$$S(\nu p) \equiv \left(\frac{\chi}{p} \right) S(\nu) + r L_{\chi,p}(m+1) \nu^{m+1} p^{m+1} \pmod{p^{m+2}}.$$

Other theorems (G.): Let $h(\nu) = S(\nu p)$. We have

$$(1) \quad h(\nu_1 \nu_2) \equiv h(\nu_1) h(\nu_2) \mod p^{m+1}, \quad gcd(\nu_1, \nu_2) = 1,$$

$$(2) \quad S(\nu_1 p) S(\nu_2) \equiv S(\nu_2 p) S(\nu_1) \mod p^{m+1}.$$

Let

$$S(N) = N^{-2} \left(\frac{-1}{4} \right)^{-\nu} \frac{(1)_\nu^5}{(\frac{1}{2})_\nu^5} \sum_{n=0}^{N-1} \frac{\left(\frac{1}{2}\right)_n^5}{(1)_n^5} \frac{(-1)^n}{4^n} (20n^2 + 8n + 1),$$

We have

$$\sum_{n=0}^{\infty} R(n) = \frac{8}{\pi^2}, \quad \text{(Ramanujan series)},$$

the extended series to the left

$$x^{-2} R_x(-) = 7 \cdot 2^6 \zeta(3) x^3 + B x^4 + C x^5 + \cdots ,$$

and the p-adic expansions

$$S(\nu p) = S(\nu) + 7 \cdot 2^6 \, \zeta_p(3) \nu^3 p^3 + B_p \nu^4 p^4 + C_p \nu^5 p^5 + \cdots .$$

We can combine S_p and S_{2p} for eliminating the first term. We get

$$S(p) - \frac{512}{99} S(2p) = \mathcal{O}(p^3),$$

In a smilar way, we obtain

$$\frac{1701}{256} S(p) + 14 S(2p) = \frac{1197}{128} + \mathcal{O}(p^4),$$

$$972 S(p) + 1024 S(2p) = 1170 - 1701 \zeta_p(3) p^3 + \mathcal{O}(p^5).$$

and

$$\frac{1767133000}{19873929} S(p) + \frac{324010496000}{1609788249} S(2p) - \frac{717225984}{1524825} S(3p) \\ - \frac{163208757248}{96994275} S(4p) = \mathcal{O}(p^5).$$

Let

$$S(N)=N^{-2}\left(\frac{-1}{80^3}\right)^{-\nu}\frac{(1)_\nu^5}{\left(\frac12\right)_\nu\left(\frac13\right)_\nu\left(\frac23\right)_\nu\left(\frac16\right)_\nu\left(\frac56\right)_\nu}$$

$$\times\sum_{n=0}^{N-1}\frac{\left(\frac12\right)_n\left(\frac13\right)_n\left(\frac23\right)_n\left(\frac16\right)_n\left(\frac56\right)_n}{(1)_n^5}(5418n^2+693n+29)\left(\frac{-1}{80^3}\right)^n.$$

We have

$$\sum_{n=0}^{\infty}R(n)=\frac{128\sqrt{5}}{\pi^2},\quad \text{(Ramanujan series)},$$

The extended series to the left

$$x^{-2}R_x(-)=42000L_5(3)x^3+Bx^4+Cx^5+\cdots,$$

and the p-adic expansions

$$S(\nu p)=\left(\frac{5}{p}\right)S(\nu)+42000L_{5,p}\nu^3p^3+B_p\nu^4p^4+\cdots.$$

$$S(N) = N^{-4} \left(\frac{1}{2^{12}} \right)^{-\nu} \frac{(1)_\nu^9}{\left(\frac{1}{2}\right)_\nu^7 \left(\frac{1}{4}\right)_\nu \left(\frac{3}{4}\right)_\nu}$$

$$\times \sum_{n=0}^{N-1} \frac{\left(\frac{1}{2}\right)_n^7 \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^9} \frac{43680n^4 + 20632n^3 + 4340n^2 + 466n + 21}{2^{12n}}.$$

we have

$$\sum_{n=0}^{\infty} R(n) = \frac{2^{11}}{\pi^4}, \quad \text{(Ramanujan series)},$$

The extended series to the left

$$x^{-4} R_x(-) = -95232\zeta(5)x^5 + Bx^6 + \cdots ,$$

and the following p-adic expansions:

$$S(\nu p) = S(\nu) - 95232\zeta_p(5)\nu^5 p^5 + B_p \nu^6 p^6 + \cdots .$$

Boris Gourevitch's series: Let

$$S(N) = \left(\frac{N^{-3}}{64}\right)^{-\nu} \frac{(1)_\nu^7}{\left(\frac{1}{2}\right)_\nu^7} \sum_{n=0}^{N-1} \frac{\left(\frac{1}{2}\right)_n^7}{(1)_n^7} \left(\frac{1}{64}\right)^n (168n^3+76n^2+14n+1).$$

We have

$$\sum_{n=0}^{\infty} R(n) = \frac{32}{\pi^3} \quad \text{(Ramanujan series).}$$

The extended series to the left

$$x^{-3} R_x(-) = 1536 L_{-4}(4)\, x^4 + B x^5 + \cdots \, ,$$

and the following p-adic identities:

$$S(\nu p) = \left(\frac{-4}{p}\right) S(\nu) + 1536 L_{-4,p}(4) \nu^4 p^4 + B_P \nu^5 p^5 + \cdots \, .$$

Let

$$U(n)=\frac{\left(\frac{1}{2}\right)_n^5\left(\frac{1}{5}\right)_n\left(\frac{2}{5}\right)_n\left(\frac{3}{5}\right)_n\left(\frac{4}{5}\right)_n}{(1)_n^9}\left(\frac{-5^5}{4^5}\right)^n.$$

$$S(N)=\frac{N^{-4}}{U(\nu)}\sum_{n=0}^{N-1}U(n)(5532n^4+5600n^3+2275n^2+42n+30).$$

The Ramanujan series $R_0(+)$ is divergent because $|z_0|>1$, but convergent by analytic continuation. We have $R_0(+)=1280/\pi^4$. The Ramanujan series $R_0(-)$ is convergent and equal to

$$\sum_{n=1}^{\infty}\frac{1}{U(n)}\frac{5532n^4-5600n^3+2275n^2-42n+30}{n^9}=-380928\zeta(5).$$

The following p-adic identities hold:

$$S(\nu p)=S(\nu)-380928\zeta_p(5)\nu^5p^5+B_P\nu^6p^6+\cdots.$$

For computational motives, we redefine $S(N)$ as

$$S(N) = \left(\prod_{i=0}^{2m} \frac{(1)_\nu}{(s_i)_\nu} \right) \sum_{n=0}^{N-1} \left(\prod_{i=0}^{2m} \frac{(s_i)_n}{(1)_n} \right) \sum_{k=0}^{m} a_k n^k z_0^n.$$

By doing it, Zudilin's conjecture generalized reads as

$$S(\nu p) \equiv \left(\frac{\chi}{p} \right) S(\nu) p^m \pmod{p^{2m+1}}, \quad \nu = 1, 2, 3, \ldots,$$

and the symmetric p-adic theorem (G.) as

$$S(\nu p) S(1) - S(\nu) S(p) \equiv 0 \pmod{p^{2m+1}}, \quad \nu = 1, 2, 3, \ldots.$$

In next examples using these kind of supercongruences, we recover the parameters a_k of the corresponding rational Ramanujan series.

Examples using

Zudilin's supercongruences generalized

We want to see that there is a series of the following form:

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^5}{(1)_n^5} \frac{(-1)^n}{4^n} (a_0 + a_1 n + a_2 n^2) = t_0 \frac{\sqrt{\chi}}{\pi^2}, \quad \chi = 1,$$

where a_0, a_1, a_2, t_0 are positive integers. Indeed, using the Wilf-Zeilberger (WZ method) we proved that $a_0 = 1, a_1 = 8, a_2 = 20$. Here, from

$$S(\nu p) - S(\nu) p^2 \equiv 0 \pmod{p^5}, \quad \nu = 1, 2, 3, \ldots,$$

and taking $p = 11$, and $\nu = 1, 2$, we get the linear system

$$103175 a_0 + 126304 a_1 + 81213 a_2 \equiv 0 \pmod{11^5},$$
$$23608 a_0 + 21777 a_1 + 22319 a_2 \equiv 0 \pmod{11^5}.$$

Let $a_0 = t$. From the above equations, we obtain

$$-66812987t - 95491225a_2 \equiv 0 \pmod{11^4},$$
$$-35044211t - 95491225a_1 \equiv 0 \pmod{11^4}.$$

Solving the equations taking into account that the inverse $\pmod{11^4}$ of 95491225 is 12252, we obtain

$$a_2 = -14621t \pmod{11^4} = 20t,$$
$$a_1 = -14633t \pmod{11^4} = 8t,$$

Hence, the solutions are of the following form:

$$a_0 = t, \quad a_1 = 8t, \quad a_2 = 20t.$$

We want to know if there is a series of the following form:

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^5} \frac{(-1)^n}{48^n} (a_0 + a_1 n + a_2 n^2) = t_0 \frac{\sqrt{\chi}}{\pi^2},$$

with $\chi = 1$, and where a_0, a_1, a_2, t_0 are positive integers. Using the PSLQ algorithm we conjecture that $a_0 = 5, a_1 = 63, a_2 = 252$ and $t_0 = 48$. Here, from

$$S(\nu p) - S(\nu) p^2 \equiv 0 \pmod{p^5}, \quad \nu = 1, 2, 3, \ldots,$$

and taking $p = 13$, and $\nu = 1, 2$, we get the linear system

$$155250 a_1 + 1838 a_2 + 327490 a_0 \equiv 0 \pmod{13^5},$$
$$304350 a_1 + 329224 a_2 + 67674 a_0 \equiv 0 \pmod{13^5}.$$

Let $a_0 = 5t$. From the above equations, we obtain

$$26628a_1 + 7535t \equiv 0 \pmod{13^4},$$
$$26628a_2 + 1579t \equiv 0 \pmod{13^4}.$$

As the inverse $\pmod{13^4}$ of 26628 is 9279, we obtain

$$a_2 = -28309t \pmod{13^4} = 252t,$$
$$a_1 = -28498t \pmod{13^4} = 63t,$$

Hence, the solutions are: $a_0 = 5t, \quad a_1 = 63t, \quad a_2 = 252t.$

We want to know if there is a series of the following form:

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^7}{(1)_n^7} \left(\frac{1}{64}\right)^n (a_0 + a_1 n + a_2 n^2 + a_3 n^3) = t_0 \frac{\sqrt{-\chi}}{\pi^3}, \quad \chi = -4,$$

where a_0, a_1, a_2, a_3, t_0 are positive integers. Using the PSLQ algorithm, we conjecture that $a_0 = 1, a_1 = 14, a_2 = 76, a_3 = 168$ and $t_0 = 16$. Here, from

$$S(\nu p) - S(\nu) \left(\frac{-4}{p}\right) p^3 \equiv 0, \pmod{p^7} \quad \nu = 1, 2, \ldots,$$

and taking $p = 11$, and $\nu = 1, 2, 3$, we get the equations

$$2078533 a_1 + 9963171 a_2 + 11695266 a_3 + 16073136 a_0 \equiv 0 \pmod{11^7},$$

$$12453192a_1 + 988367a_2 + 3883033a_3 + 14086913a_0 \equiv 0 \pmod{11^7},$$
$$17113786a_1 + 2247378a_2 + 4011161a_3 + 7012796a_0 \equiv 0 \pmod{11^7}.$$

Let $a_0 = t$. From the above equations, we obtain

$$7854385a_1 + 3429250a_2 + 19159030t \equiv 0 \pmod{11^4},$$
$$3851936a_1 + 8961898a_2 + 5481146t \equiv 0 \pmod{11^4}.$$

Solving the equations, we obtain

$$a_1 = -11965t \pmod{11^4} = 14t,$$
$$a_2 = -1255t \pmod{11^4} = 76t,$$
$$a_3 = -14473t \pmod{11^4} = 168t.$$

Examples using the

symmetric p-adic theorem (G.)

We want to know if there is a series of the following form:

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^5} \frac{1}{7^{4n}} (a_0 + a_1 n) = t_0 \frac{\sqrt{-\chi}}{\pi}, \quad \chi = -3,$$

where a_0 and a_1 are positive integers, and t_0 is a rational. Using the theory of modular functions it was proved that $a_0 = 3$ and $a_1 = 40$. Here, we will prove it from the theorem

$$S(\nu p)S(1) - S(\nu)S(p) \equiv 0 \pmod{p^3}, \quad \nu = 1, 2, 3, \ldots.$$

For that aim, we let $a_0 = 3t$, take $p = 23$, and use the equations for $\nu = 1, 2$, namely

$$4163a_1^2 + 9108a_1 t + 7406a_0^2 \equiv 0 \pmod{23^3},$$
$$7682a_1^2 + 2185a_1 t + 7406a_0^2 \equiv 0 \pmod{23^3}.$$

Substracting both equations and dividing by a_1, we obtain

$$8648a_1 + 6923t \pmod{23^3}.$$

Simplifying by 23, we get

$$376a_1 + 301t \pmod{23^2}.$$

The inverse of 376 $\pmod{23^2}$ is 325. Therefore, multiplying by 325, we see that

$$a_1 + 489t = 0 \pmod{23^2}.$$

Hence

$$a_1 = -489t = 40t \pmod{23^2},$$

and the solution is $a_0 = 3t$ and $a_1 = 40t$.

We want to know if there is a series of the following form:

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^7 \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^9} \frac{1}{2^{12n}} (a_0 + a_1 n + a_2 n^2 + a_3 n^3 + a_4 n^4) = t_0 \frac{\sqrt{\chi}}{\pi^4},$$

where χ is the character (an integer), a_k are positive integers, and t_0 is a rational. In 2010, Jim Cullen using PSLQ conjectured that $a_0 = 21, a_1 = 466, a_2 = 4340, a_3 = 20632, a_4 = 43680$ with $\chi = 1$ and $t_0 = 2048$. Here, we will prove it from the theorem

$$S(\nu p)S(1) - S(\nu)S(p) \equiv 0 \pmod{p^9}, \quad \nu = 1, 2, 3, \ldots.$$

Indeed, let $p = 7$ (a prime), and $a_0 = 3t$, $a_1 = 466t$, $a_2 = 4340t$, and $a_3 = 20632t$.

Taking $\nu = 2, 3$, we get the equations

$$23785306a_4^2 + 35827295a_4t + 20891591t^2 \equiv 0 \pmod{7^9},$$
$$2555244a_4^2 + 35104587a_4t + 18959962t^2 \equiv 0 \pmod{7^9}.$$

From the above system we can eliminate a_4^2, and we obtain

$$410780a_4 + 2955113t \equiv 0 \pmod{7^8}.$$

The inverse of $410780 \pmod{7^8}$ is 531586, and finally we obtain

$$a_4 = -5721121t = 43680t \pmod{7^8},$$

which is the correct integer value of a_4.

I am very grateful to Wadim Zudilin for sharing several important ideas on the p adics, and very specially for advising me to replace x with $p, 2p, 3p, \dots,$ and not only with p.

THANK YOU.