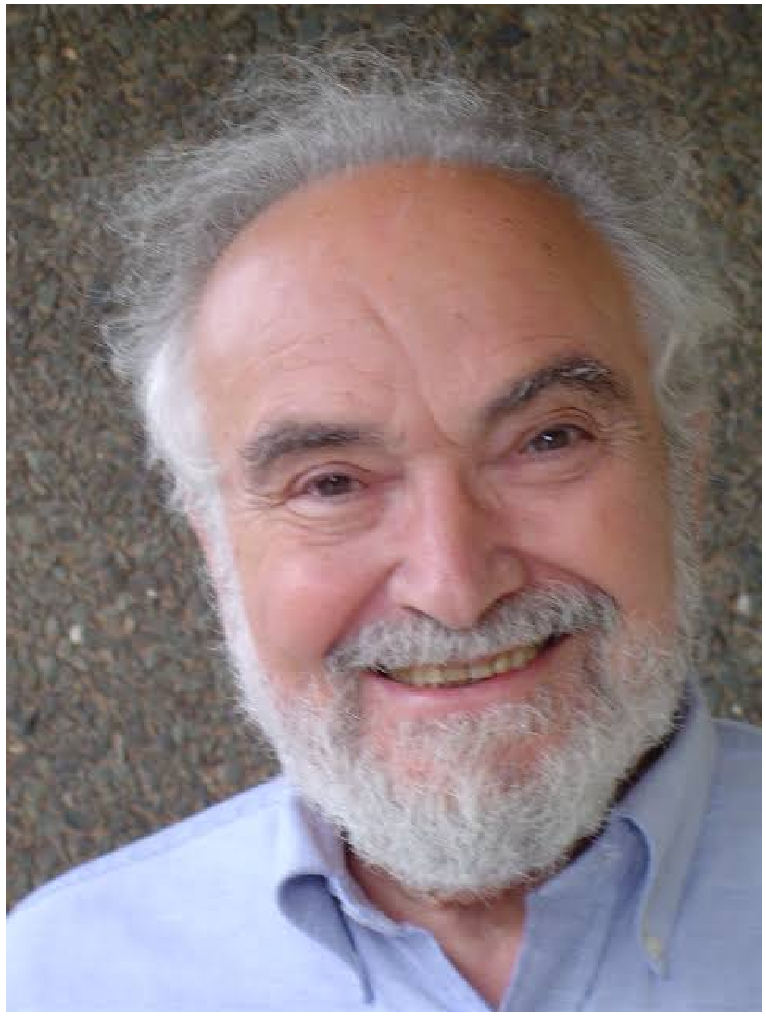
The Mathematics and Physics of Joel Lebowitz

RUTGERS EXPERIMENTAL MATHEMATICS SEMINAR

May 8, 2025

Abstract: Joel Lebowitz (b. May 10, 1930) has made many important contributions to both mathematics and physics. Some of them will be outlined in this talk.



According to ResearchGate, Joel has 896 publications . . . a total of 562 articles, 38 editorials, 19 commentaries, 8 review articles.

Mathematical Physics

Mathematics is easy.

Physics is hard.

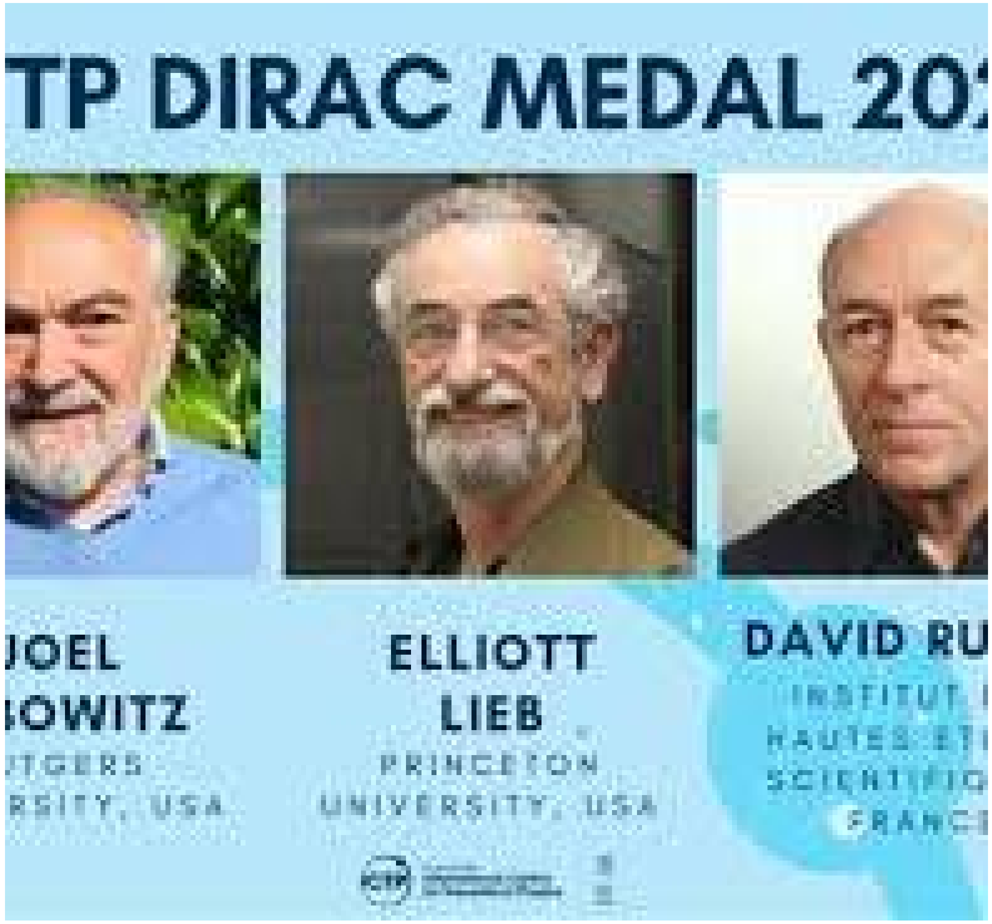
Statistical Mechanics

small $\rightarrow$ LARGE

The Soul of Statistical Mechanics.

a historical role in the development of modern mathematical physics. ... in the case of Joel Lebowitz one can say that, for several decades, he has been the soul of statistical mechanics.

David Ruelle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Statistical Mechanics, small

Mechanics: $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_N$, $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_N$, $H(q,p)$

$$d\boldsymbol{q}_i/dt = \partial H/\partial \boldsymbol{p}_i, \qquad d\boldsymbol{p}_i/dt = -\partial H/\partial \boldsymbol{q}_i$$

$$H(q,p) = \sum_i \frac{\boldsymbol{p}_i^2}{2m_i} + V(q)$$

Statistical Mechanics, large

- Thermodynamics: U, T, p, V, ρ, S . . .

  Energy, . . . , Entropy . . .

- Statistical: LLN

Statistical Mechanics

- Equilibrium: “easy”

- Nonequilibrium: “hard”

Equilibrium Statistical Mechanics

$$\mu \quad \longleftrightarrow \quad \frac{1}{Z} e^{-\beta H} \qquad \left(\beta = \frac{1}{kT}\right)$$

$$\log Z(T, N, \Lambda, \ldots)$$

Thermodynamic Limit

$$\lim_{V \to \infty} \frac{1}{V} \log Z \qquad (V = |\Lambda|)$$

Coulomb Systems

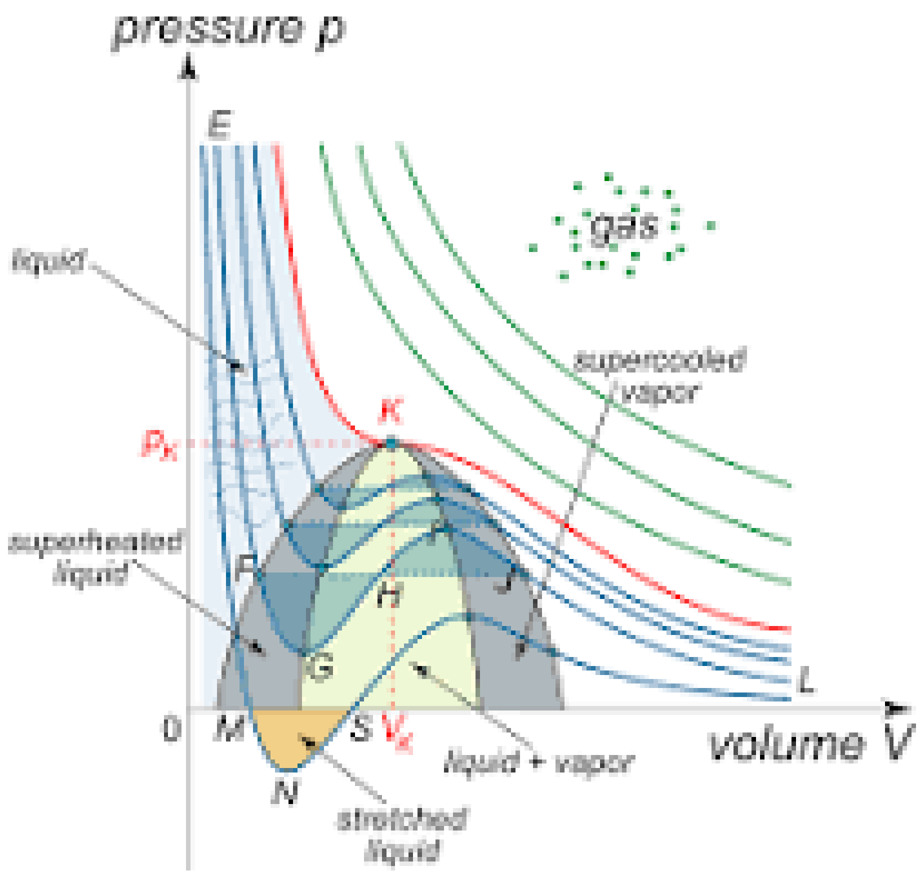
Phase Transitions

Equations of State:

- ideal gas: $pv = kT$

- van der Waals: $(p + \frac{a}{v^2})(v - b) = kT$

Phase Diagram



Joel with Oliver Penrose

Lebowitz Inequalities

$$\langle\sigma_i\sigma_j\sigma_k\sigma_l\rangle - \langle\sigma_i\sigma_j\rangle\langle\sigma_k\sigma_l\rangle \leq \langle\sigma_i\sigma_k\rangle_c\langle\sigma_j\sigma_l\rangle_c + \text{one more}$$

Nonequilibrium Statistical Mechanics

- μ ?????

- approach to equilibrium

Nonequilibrium Statistical Mechanics

- NESS

- dynamics

NESS, Heat Flow and Fourier's Law

- Joel and Herbert Spohn

- Joel and Stefano Olla

Dynamics: Questions

- Why do systems approach equilibrium?

- How do systems approach equilibrium? (evolution equations?)

- (What is thermal equilibrium?)

More Questions: Puzzles and Paradoxes

- irreversibility? $\left(md^2Q(t)/dt^2 = F(Q)\right)$

- Second Law: $dS/dt \geq 0$

- What is entropy?

Joel in *Physics Today*

- early Joel: "Modern Ergodic Theory" (February 1973, with Oliver Penrose)

- later Joel: "Boltzmann's Entropy and Time's Arrow?" (September 1993)

Boltzmann's Equation

At low density, typically,

$$f_{X(t)}(\mathbf{q}, \mathbf{v}) \approx f_t(\mathbf{q}, \mathbf{v})$$

where f_t obeys Boltzmann's equation

$$\frac{\partial f_t}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{q}} f_t = Q(f_t)$$

Consequences

- $f_t(\mathbf{q},\mathbf{v}) \to f_{eq}(\mathbf{q},\mathbf{v}) \sim e^{-\frac{1}{2}m\mathbf{v}^2/kT}$

- $H(f_t) = \int f_t(\mathbf{q},\mathbf{v}) \log f_t(\mathbf{q},\mathbf{v}) \mathbf{dqdv}$ $\searrow$ as t $\nearrow$

Entropy: $S(X) = -NH(f_X)$

Paradoxes

- Loschmidt: irreversibility

- Zermelo: Poincaré recurrence

Poincaré

The kinetic theory of gases is up to now the most serious attempt to reconcile mechanism and experience, but it is still faced with the difficulty that a mechanical system cannot tend toward a permanent final state but must always return eventually to a state very close to its initial state [recurrence]. This difficulty is overcome only if one is willing to assume that the universe does not tend irreversibly to a final state, as seems to be indicated by experience, but will eventually regenerate itself and reverse the second law of thermodynamics.

Mechanism and Experience

What is Boltzmann’s equation about?

What is Entropy?

Entropy vs Energy

A Variety of Notions of Entropy

- Thermodynamic entropy
  - equilibrium entropy
  - non-equilibrium entropy
- Boltzmann entropy
- Gibbs entropy
- Shannon entropy

- Kolmogorov-Sinai entropy
- Tsallis entropy
- relative entropies
- von Neumann entropy
- von Neumann entropy'
- quantum Boltzman entropy
- diagonal entropy

Gibbs Entropy

$$S_G(\varrho) = -\int \varrho(X) \log \varrho(X) dX$$

$$\varrho(X) \equiv \varrho(\mathbf{q}_1, \mathbf{v}_1, \ldots, \mathbf{q}_N, \mathbf{v}_N)$$

Gibbs Entropy and the H-function

$$H(f) = \int f(\mathbf{q}, \mathbf{v}) \log f(\mathbf{q}, \mathbf{v}) \mathbf{dqdv}$$

low density: $\varrho(X) \sim \prod_i f(\mathbf{q}_i, \mathbf{v}_i)$

$$S_G(\varrho) = -NH(f)$$

More Paradox

$S_G(\varrho_t)$ does not change with t!

(for a closed system)

I think that Boltzmann's idea is staggering in its boldness and beauty. But I also think that it is quite untenable, at least for a realist. It brands unidirectional change as an illusion. This makes the catastrophe of Hiroshima an illusion. Thus, it makes our world an illusion and with it all our attempts to find out more about the world. (Karl Popper)

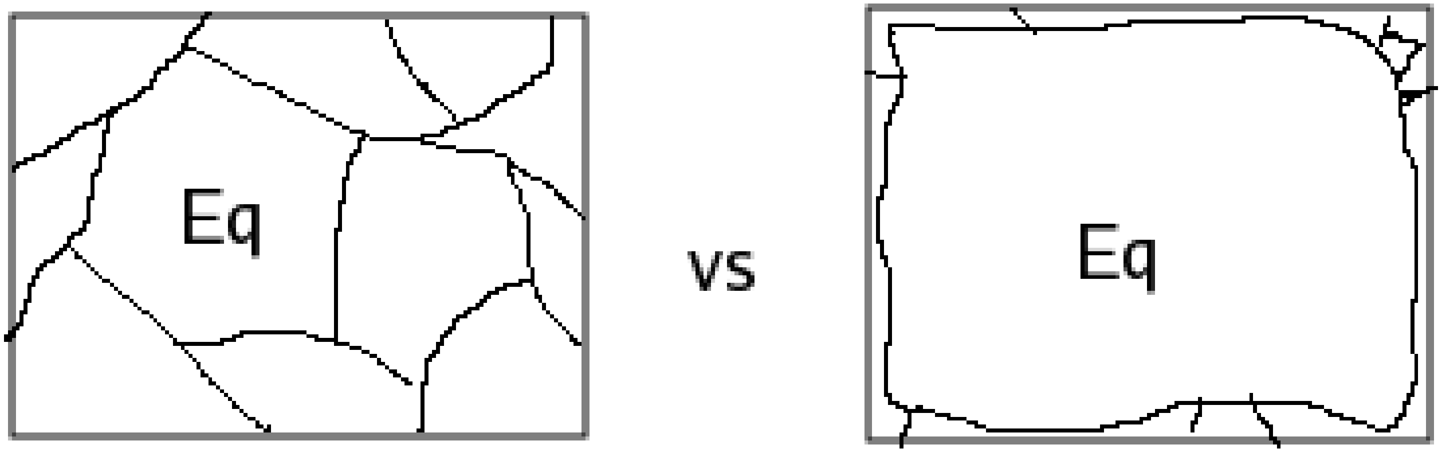
The time variations of the entropy are then based on the fact that the observer does not know everything, that he cannot find out (measure) everything which is measurable in principle.
(John von Neumann)

The spontaneous transition from order to disorder is the quintessence of Boltzmann's theory . . . This theory really grants an understanding and does not . . . reason away the dissymetry of things by means of an a priori sense of direction of time . . . No one who has once understood Boltzmann's theory will ever again have recourse to such expedients. It would be a scientific regression beside which a repudiation of Copernicus in favor of Ptolemy would seem trifling. (Schrödinger)

- What is thermal equilibrium?

- What is thermodynamic entropy?

Macrovariables and Macrostates



$$\frac{|\Gamma_{eq}|}{|\Gamma|} \approx 1$$

system is in equilibrium $\leftrightarrow X \in \Gamma_{eq}$

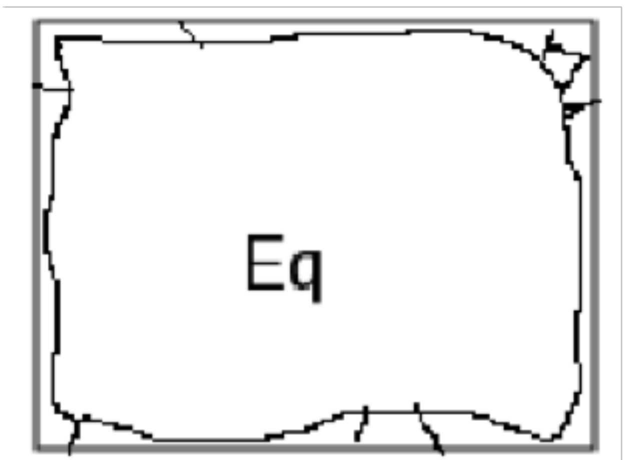
Boltzmann Entropy

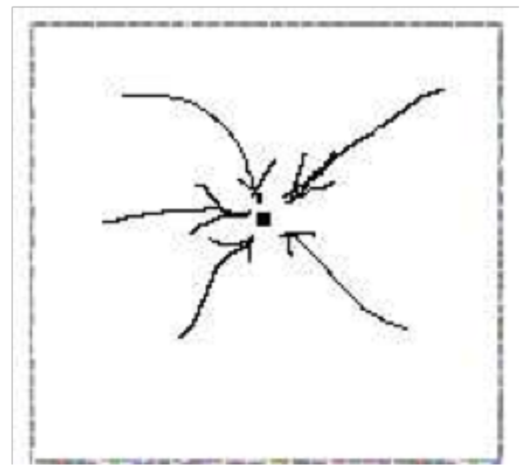
$$S_B(X) = \log |\mathit{\Gamma}_{f_X}|$$

$$S_B(X) = -NH(f_X)$$

$\big($$S_B(X) = S_G(\varrho)$ when ϱ is uniform on $\mathit{\Gamma}_{f_X}$.)

$\Gamma_{f_{eq}}$ $\qquad\qquad$ f_{eq}





$X \in \Gamma_E$ $\qquad$ f_X $\qquad$ $f -$ space

microscopic picture $\qquad\qquad$ macroscopic picture

One should not forget that the Maxwell distribution is not a state in which each molecule has a definite position and velocity, and which is thereby attained when the position and velocity of each molecule approach these definite values asymptotically. . . . It is in no way a special singular distribution which is to be contrasted to infinitely many more non-Maxwellian distributions; rather it is characterized by the fact that by far the largest number of possible velocity distributions have the characteristic properties of the Maxwell distribution, and compared to these there are only a relatively small number of possible distributions that deviate significantly from Maxwell's. Whereas Zermelo says that the number of states that finally lead to the Maxwellian state is small compared to all possible states, I assert on the contrary that by far the largest number of possible states are "Maxwellian" and that the number that deviate from the Maxwellian state is vanishingly small.

I have . . . emphasized that the second law of thermodynamics is from the molecular viewpoint merely a statistical law. Zermelo's paper shows that my writings have been misunderstood; . . . Poincaré's theorem, which Zermelo explains at the beginning of his paper, is clearly correct, but his application of it to the theory of heat is not. . . . Thus, when Zermelo concludes, from the theoretical fact that the initial states in a gas must recur – without having calculated how long a time this will take – that the hypotheses of gas theory must be rejected or else fundamentally changed, he is just like a dice player who has calculated that the probability of a sequence of 1000 one's is not zero, and then concludes that his dice must be loaded since he has not yet observed such a sequence! (Boltzmann)

The applicability of probability theory to a particular case cannot of course be proved rigorously. . . . Despite this, every insurance company relies on probability theory. . . . It is completely incomprehensible to me how anyone can see a refutation of the applicability of probability theory in the fact that some other argument shows that exceptions must occur now and then over a period of eons of time; for probability theory itself teaches just the same thing. (Boltzmann)

Quantum Mechanics

- $\cdots$

- Large Deviations

1. *Boltzmann's entropy and time's arrow,* Physics Today, 1993, Cited by 661

2. *Statistical mechanics: A selective review of two central issues,* Reviews of Modern Physics, 1999, Cited by 289

3. *A new algorithm for Monte Carlo simulation of Ising spin systems,* with A.B. Bortz and M.H. Kalos, Journal of Computational Physics, 1975, Cited by 3049

4. *Canonical typicality,* with S. Goldstein, R. Tumulka, N. Zanghi, Physical Review Letters, 2006, Cited by 905

5. *Modern ergodic theory,* with O. Penrose, Physics Today, 1973, Cited by 326

6. *Macroscopic laws, microscopic dynamics, time's arrow and Boltzmann's entropy,* Physica A: Statistical Mechanics and its Applications, 1993, Cited by 277

7. *New approach to nonequilibrium processes,* with P.G. Bergmann, Physical Review, 1955, Cited by 276

8. *Time symmetry in the quantum process of measurement,* with Y. Aharonov and P.G. Bergmann, Physical Review, 1964, Cited by 1230

9. *Statistical mechanics of the nonlinear Schrödinger equation,* with H.A. Rose and E.R. Speer, Journal of Statistical Physics, 1988, Cited by 396

10. *A Gallavotti-Cohen-type symmetry in the large deviation functional for stochastic dynamics,* with H. Spohn, Journal of Statistical Physics, 1999, Cited by 1771

Joel

The great insight of Boltzmann was . . . to identify the entropy of a macroscopic system in a specified macrostate with the logarithm of the “number” . . . of the microstates corresponding to that macrostate.

. . . no perception in physics has ever seemed more important to me than that of Boltzmann—despite Planck and Einstein. (Schrödinger)

The End

What is thermal equilibrium?

Two views, corresponding to two different attitudes towards the foundations of statistical mechanics:

- individualist

- ensemblist

Individualist

A system is in thermal equilibrium if it is in an appropriate pure state, given by a point in phase space.

Ensemblist

A system is in thermal equilibrium if it is in an appropriate statistical state, given by a probability measure on phase space.

Equilibrium: individualist

$$X = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_N, \boldsymbol{p}_1, \ldots, \boldsymbol{p}_N)$$

$$\Gamma = \{X : H(X) = E\}$$

$$\Gamma = \bigcup_{\nu} \Gamma_\nu$$

Equilibrium: ensemblist

A system is in equilibrium if its state X is random, with distribution

$$\rho = \rho_{mc} \quad \text{or} \quad \rho = \rho_{can} = e^{-\beta H}/Z$$

Approach to equilibrium: ensemblist

$$\rho_t \longrightarrow \rho_{mc} \text{ (or } \rho_{can}\text{) as } t \to \infty$$

mixing . . . a possibility

Approach to equilibrium: individualist

$$X_t \in \Gamma_{eq} \text{ (or near } \Gamma_{eq}\text{) as } t \to \infty$$

is typically impossible. Poincaré recurrence.

It will typically not be the case that the system is in, or near, equilibrium for all sufficiently large times.

“Approach” to equilibrium (individualist):

$$X_t \in \Gamma_{eq}$$

for *most* (sufficiently large) t (even when the system is initially not in equilibrium).

1865 Clausius, entropy

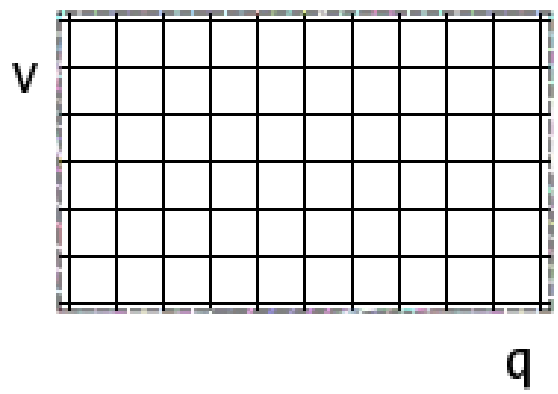
1867 Maxwellian velocities

$$\rho_{eq}(\mathbf{v}) \sim e^{-\frac{1}{2}m\mathbf{v}^2/kT}$$

1872 Boltzmann's equation

1877 Entropy and equilibrium macrostates

1872: $X = (\mathbf{q}_1, \mathbf{v}_1, \ldots, \mathbf{q}_N, \mathbf{v}_N)$



$$f_{emp}(\mathbf{q}, \mathbf{v}) \equiv f_X(\mathbf{q}, \mathbf{v})$$

$$f_X(\mathbf{q},\mathbf{v}) = \frac{|X \cap \boldsymbol{\Delta}(\mathbf{q},\mathbf{v})|}{|\boldsymbol{\Delta}(\mathbf{q},\mathbf{v})|N}$$

$$= \frac{n_X(\boldsymbol{\Delta}(\mathbf{q},\mathbf{v}))/N}{|\boldsymbol{\Delta}(\mathbf{q},\mathbf{v})|}$$

$$f_{emp}(\mathbf{q},\mathbf{v},t) \equiv f_{X(t)}(\mathbf{q},\mathbf{v})$$

1877: Macrostates

$$\Gamma_f = \{X \in \Gamma_E \,|\, f_X(\mathbf{q}, \mathbf{v}) \approx f(\mathbf{q}, \mathbf{v})\}$$

1877$\leftrightarrow$1872: At low density

$$|\Gamma_f| \sim e^{-NH(f)}$$

$N \sim 10^{20}$: most of Γ_E is $\Gamma_{f_{eq}}$

$$\log |\Gamma_f| = -NH(f)$$

Smallness of Atypical Events

$$10^{-10^{20}} \text{ (for } N = 10^{20}\text{)}$$

"Typically"

By far most solutions $X(t)$ starting in $\Gamma_0 = \Gamma_{f_0}$

typicality $\leftrightarrow \mu_{\Gamma_0}$

Note: Not μ_{Γ_t} ($\Gamma_t = \Gamma_{f_t}$), even when we are concerned with the continuation beyond time t!

The "Hard" Problem

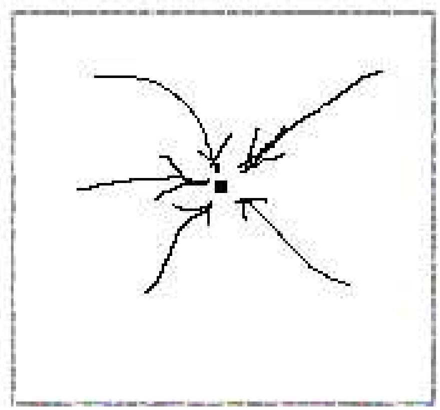
So we had to keep waiting until we were alive to notice it—we had to have at least that big a fluctuation. But I believe this theory to be incorrect. I think it is a ridiculous theory for the following reason. . . . In fact if the thing were a fluctuation, and I noticed something odd, the most likely way that it got there would be that there was nothing odd anywhere else. . . . And since we always make the prediction that in a place where we have not looked we shall see stars in a similar condition, or find the same statement about Napoleon, or that we shall see bones like the bones that we have seen before, the success of all those sciences indicates that the world did not come from a fluctuation . . . Therefore I think it is necessary to add to the physical laws the hypothesis that in the past the universe was more ordered . . . than it is today—I think this is the additional statement that is needed to make sense, and to make an understanding of the irreversibility. (Feynman, *The Character of Physical Law*)

The second law of thermodynamics can be proved from the mechanical theory if one assumes that the present state of the universe, or at least that part which surrounds us, started to evolve from an improbable state and is still in a relatively improbable state. This is a reasonable assumption to make, since it enables us to explain the facts of experience, and one should not expect to be able to deduce it from anything more fundamental.

In order to produce a universe resembling the one in which we live, the Creator would have to aim for an absurdly tiny volume of the phase space of possible universes – about $1/10^{10^{123}}$ of the entire volume, for the situation under consideration. [The pin, and the spot aimed for, are not drawn to scale!] (R. Penrose, *The Emperor's New Mind*)

$\Gamma_{f_{eq}}$ $\qquad\qquad$ f_{eq}

$\rightarrow$



$X \in \Gamma_E \qquad f_X \quad f - \text{space}$

microscopic picture $\qquad$ macroscopic picture