A Functional Approach to Graph Labelings

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Incorporates insights from joint works with

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Leslie Hogben, Michael Williams, Richard Low

Graph Decomposition Problem

A decomposition of a graph G is a list $[G_k : 0 \leq k < m]$ of edge disjoint subgraphs of G i.e.

$$G = \bigcup_{0 \leq k < m} G_k \text{ such that } \begin{cases} \emptyset = E(G_i) \cap E(G_j) \\ \text{for all } 0 \leq i < j < m \end{cases}.$$

In the example below $K_4 = G_0 \cup G_1$ such that $\emptyset = E(G_0) \cap E(G_1)$

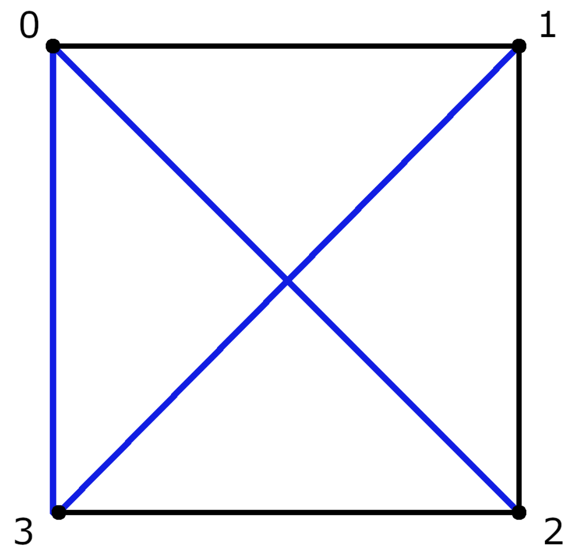
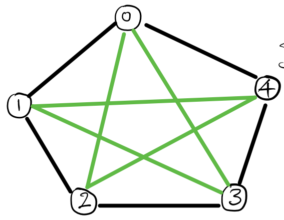
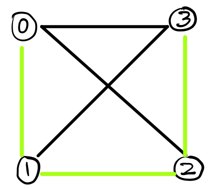


Figure: Decomposition of K_4 into copies of P_4

More Examples

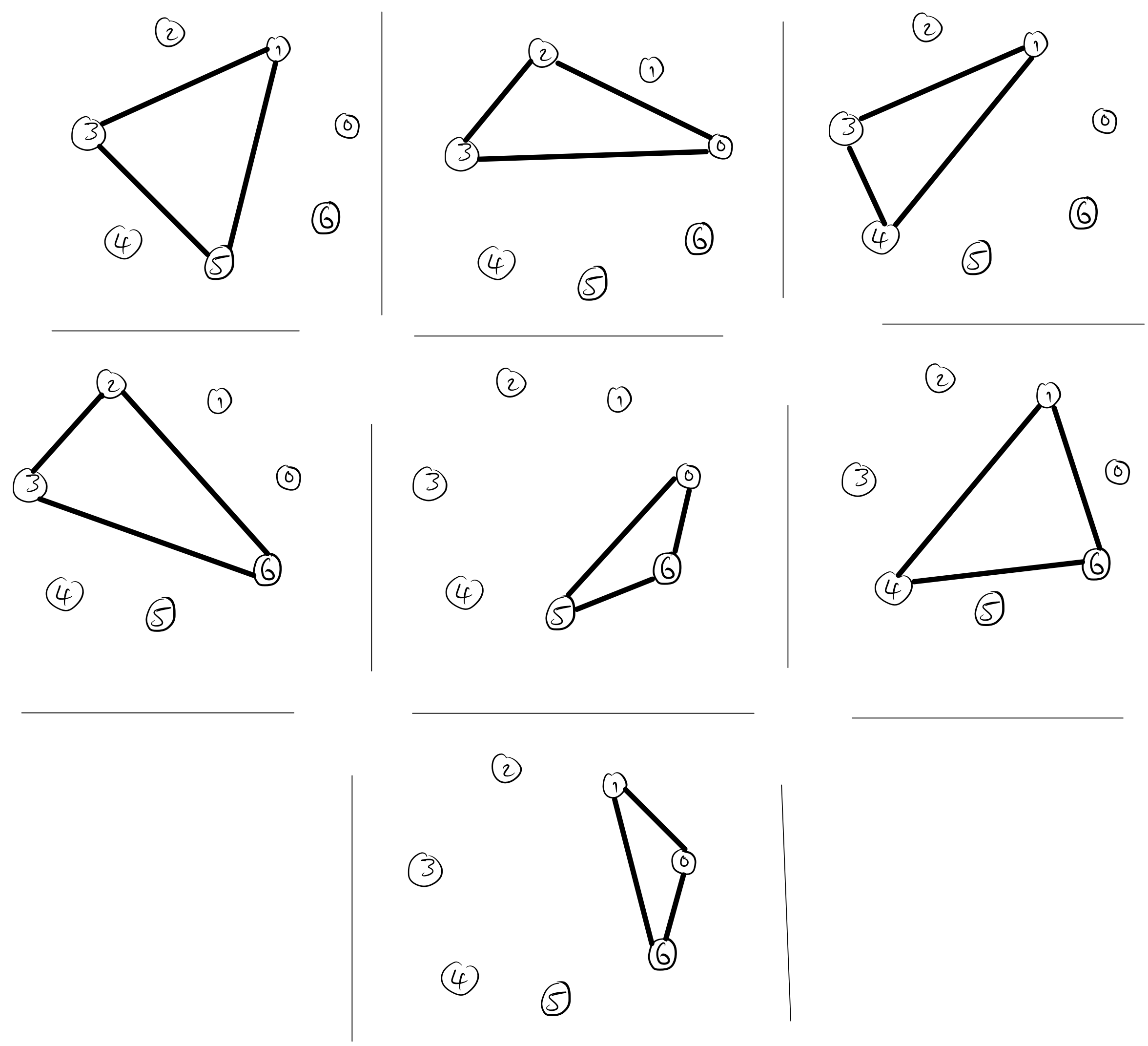


Decomposition of K_5 into 2 copies of C_5



Decomposition of K_4 into 2 copies of P_4

Decomposition of K_7 into copies of K_3.

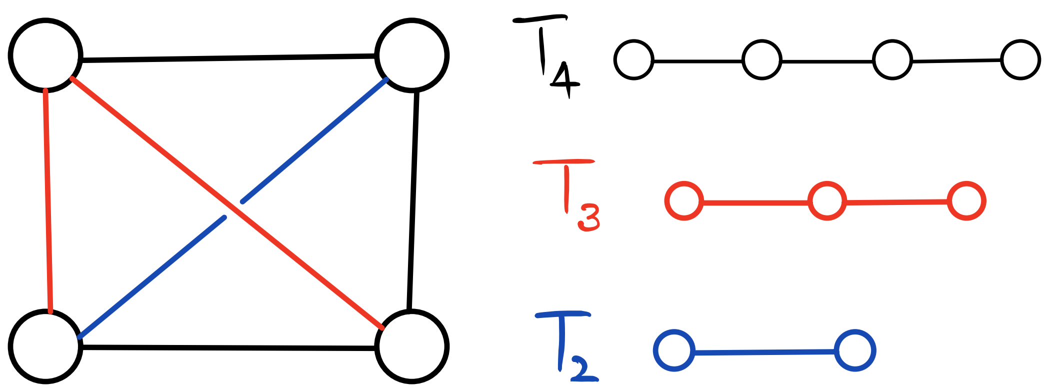


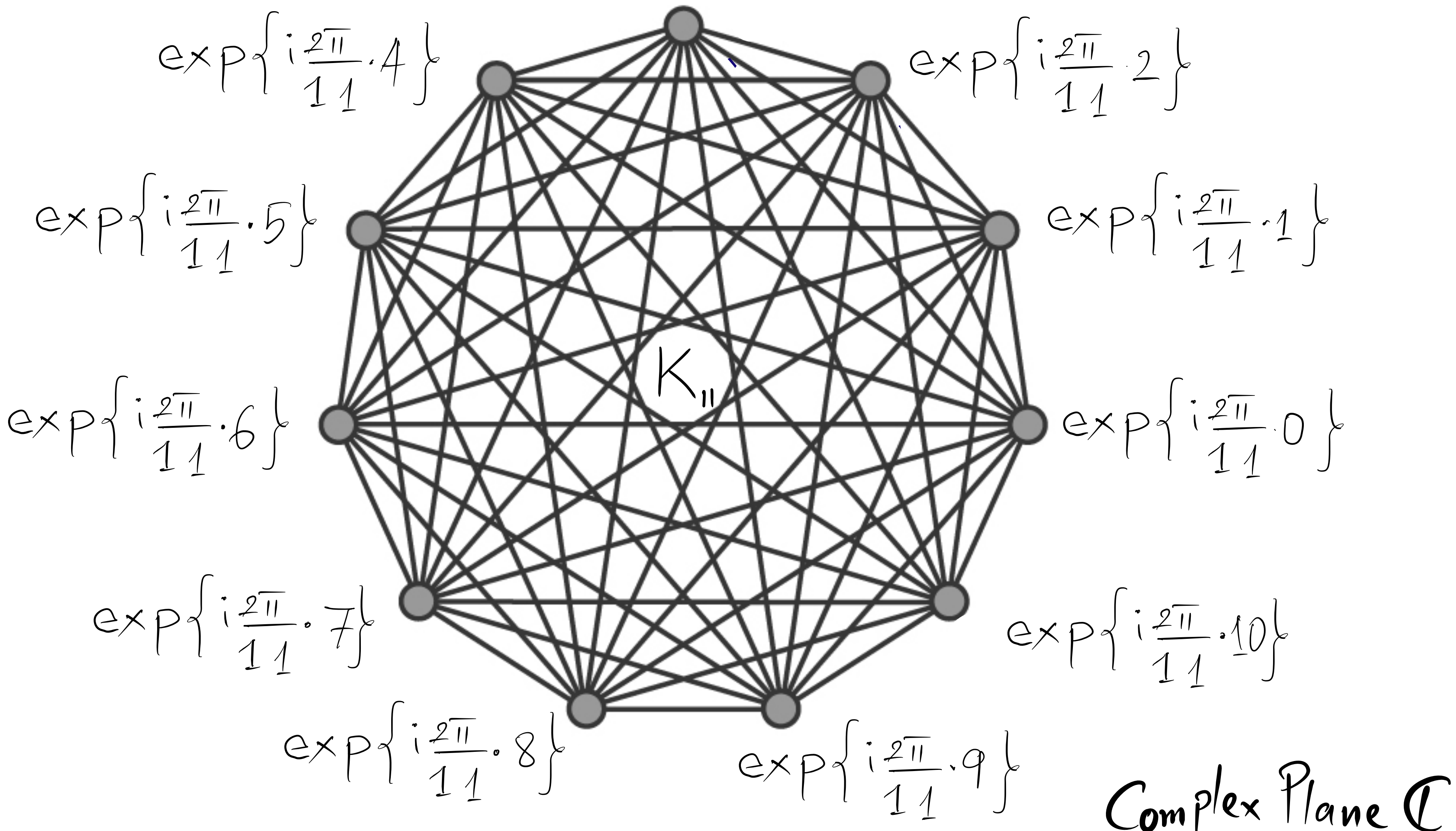
Three Tree Decomposition Problems

Problem 1: Gyárfás-Lehel Conjecture (1976)

Conjecture 1 (TPC)**.** *For $2 \leq i \leq n$, let T_i be a tree on i vertices. Then the set of trees $T_2, \ldots, T_n$ has a packing into the complete graph on n vertices.*

Example : K_4 decomposes into




K11
exp{i 2π/11 · 0}
exp{i 2π/11 · 1}
exp{i 2π/11 · 2}
exp{i 2π/11 · 4}
exp{i 2π/11 · 5}
exp{i 2π/11 · 6}
exp{i 2π/11 · 7}
exp{i 2π/11 · 8}
exp{i 2π/11 · 9}
exp{i 2π/11 · 10}
Complex Plane ℂ

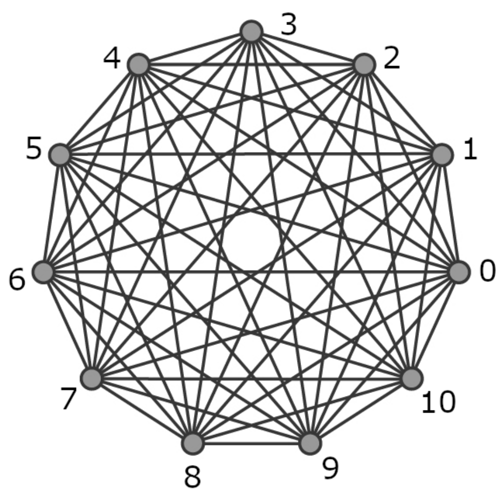
The Kotzig–Ringel–Rosa conjecture

KRR claim:

K_{2n-1} cyclically decomposes into edge disjoint copies of any tree on n "consecutive" vertices.

$$K_{2n-1} = \bigcup_{0 \le k < 2n-1} \sigma^{(k)} T \sigma^{(-k)} \text{ such that } \sigma = \text{id} + 1 \bmod 2n-1,$$

$$\emptyset = E\left(\sigma^{(i)} T \sigma^{(-i)}\right) \cap E\left(\sigma^{(j)} T \sigma^{(-j)}\right), \quad 0 \le i < j < 2n-1.$$



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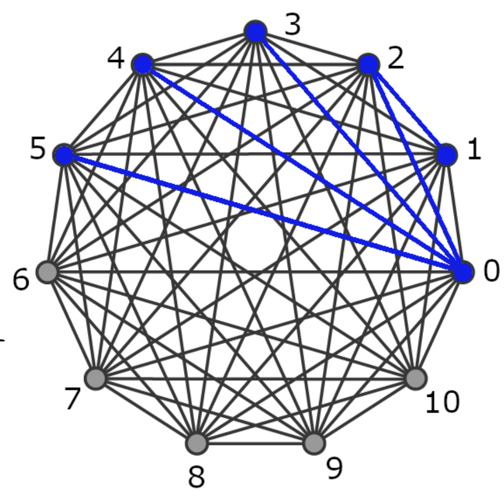
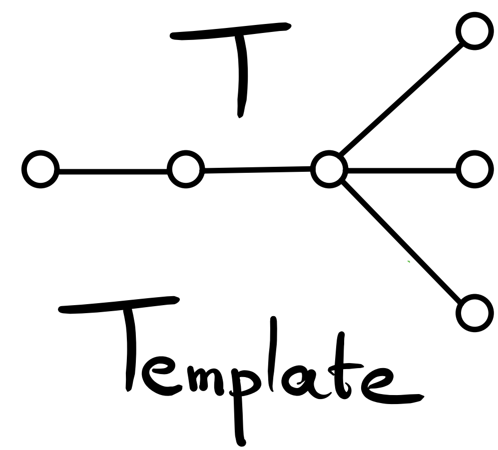
Place the template

Perform the transformation

$$z \mapsto \exp\left\{\frac{2\pi\sqrt{-1}}{11}\right\} \cdot z$$





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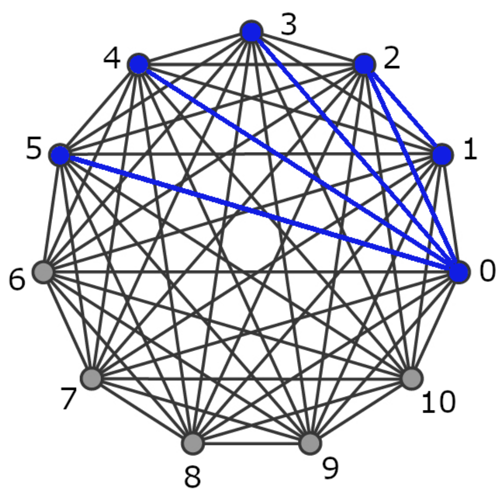

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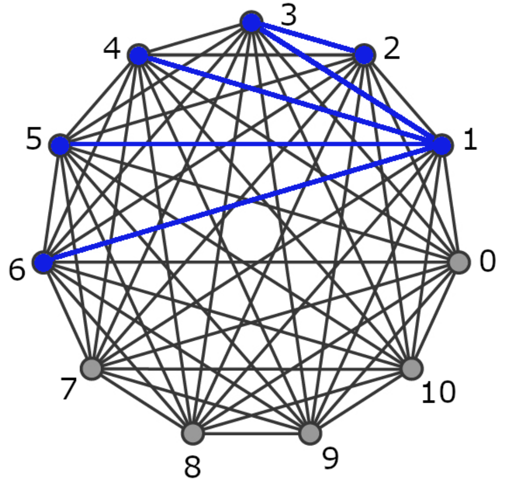

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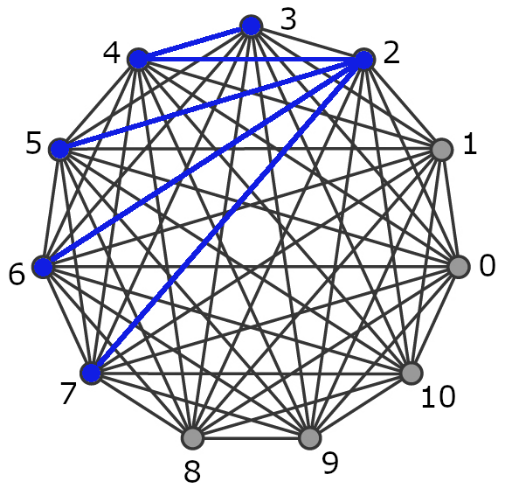

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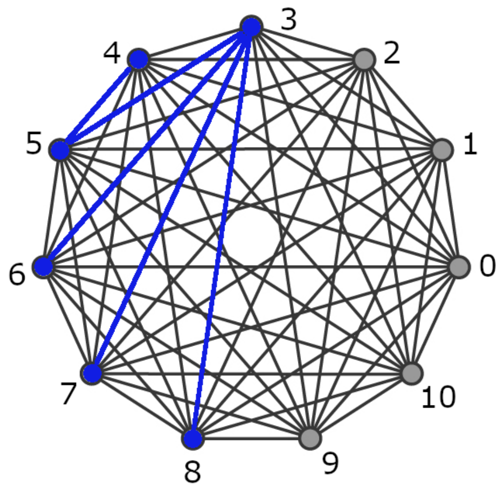

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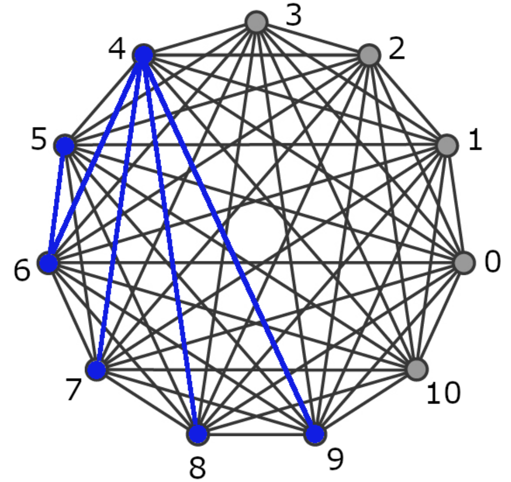

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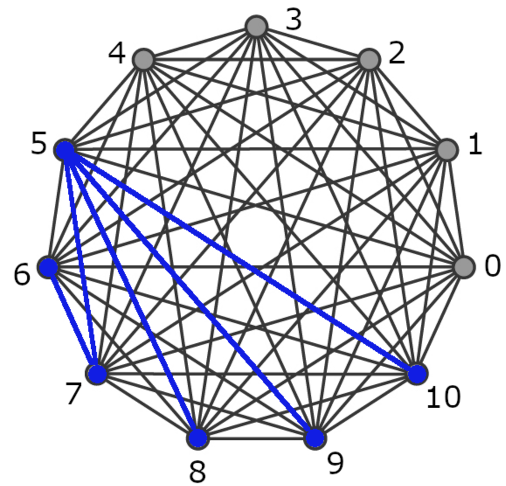

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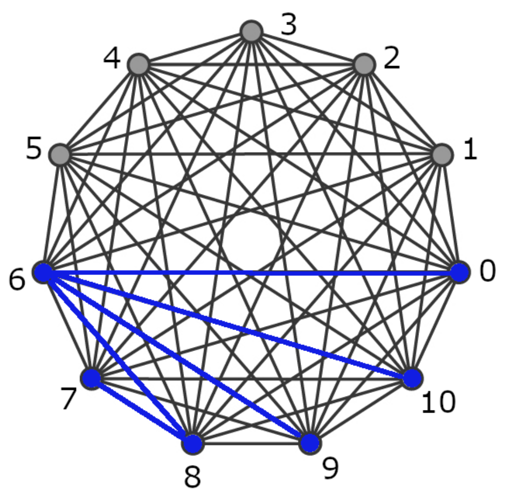

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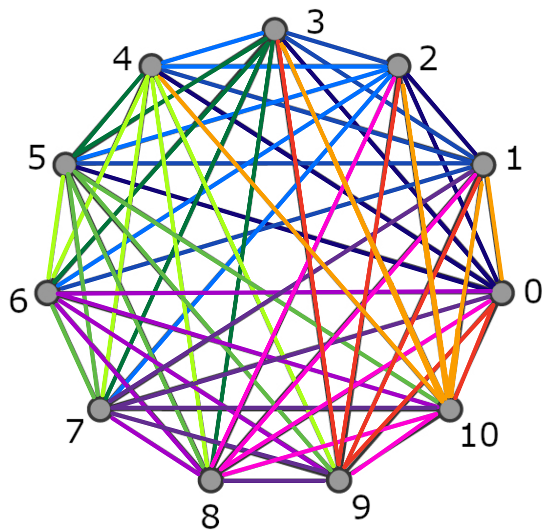
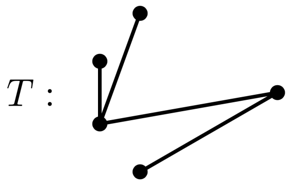


The Ringel Conjecture

Theorem (2020) : Asymptotic Decomposition Result

K_{2n-1} decomposes into edge disjoint copies of any tree on n vertices.

By a proof of Richard Montgomery, Alexey Pokrovskiy, Benny Sudakov and independently by Peter Keevash, Katherine Staden



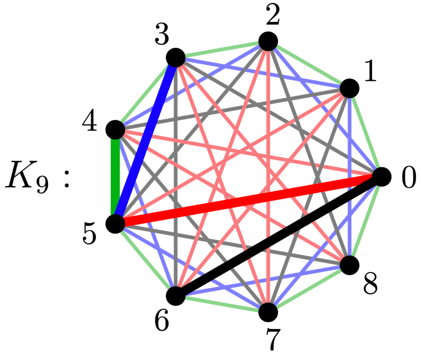


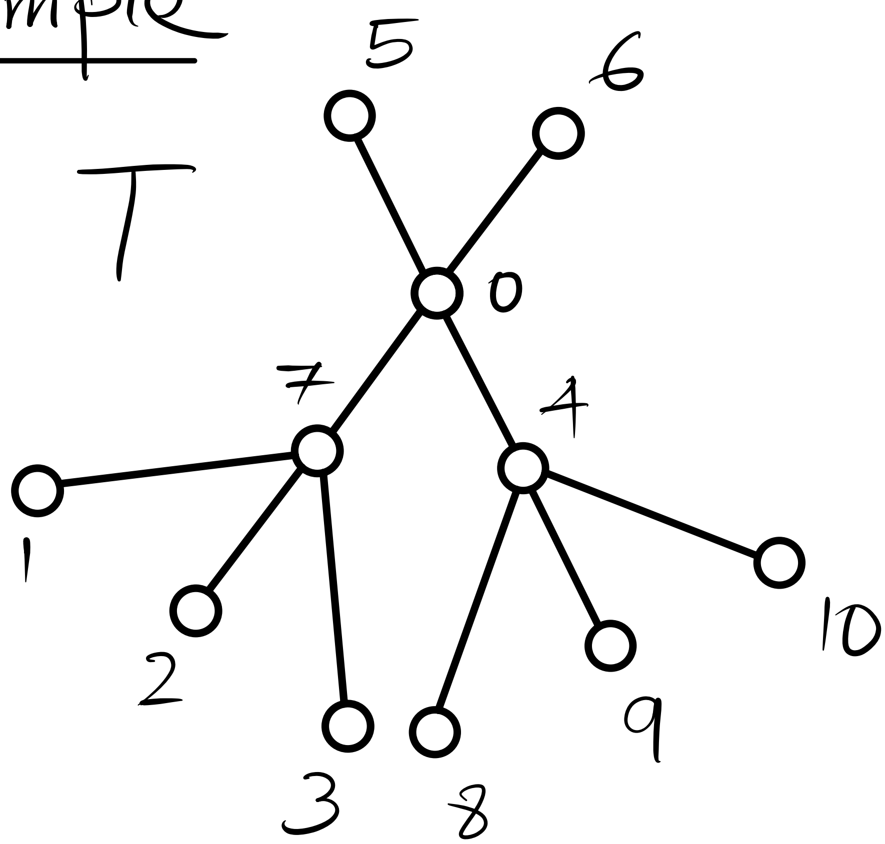
Figure: Decomposition of K_9 into copies of C_5

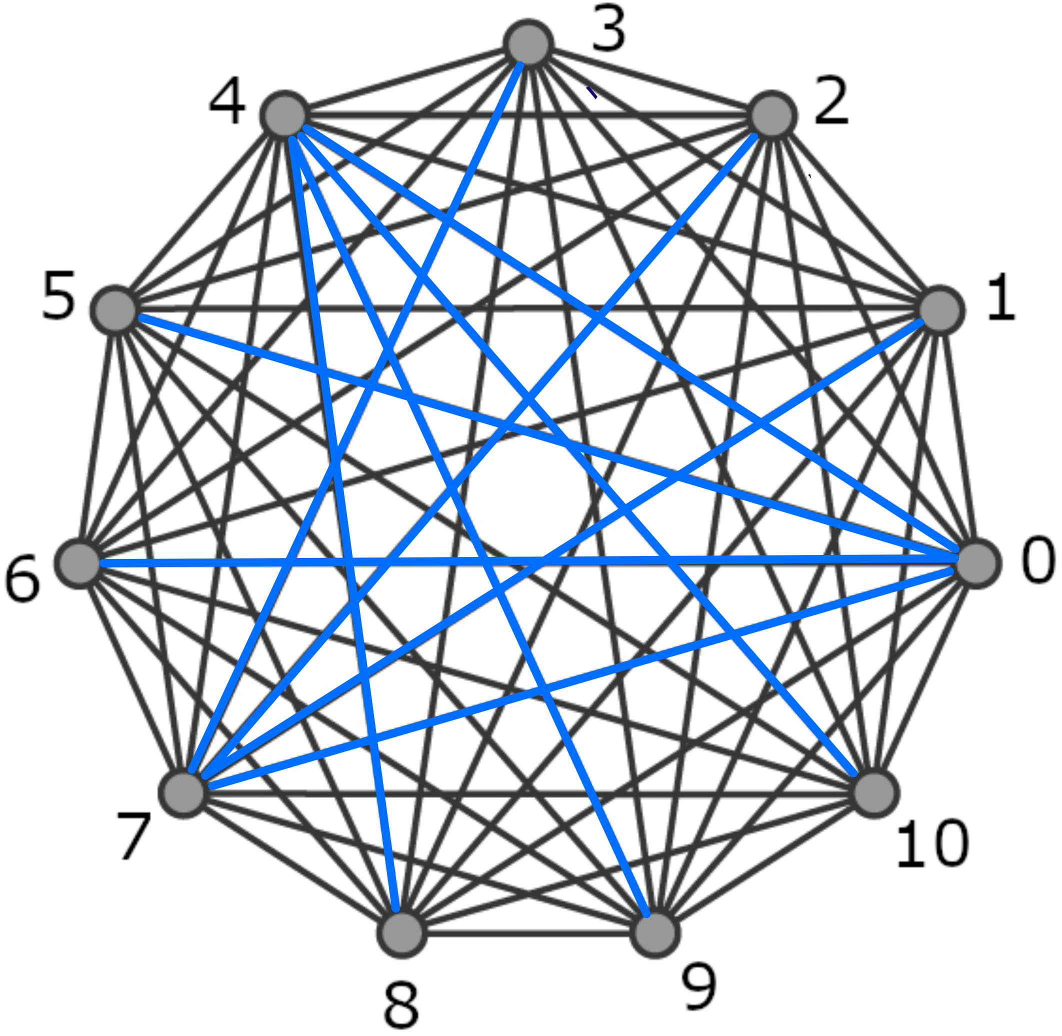
Graham–Sloane Conjecture (1980)

Every n-vertex tree admits an embedding into K_n such that no two edges are parallel.

Example

Given T




3
2
4
5
1
6
0
7
10
8
9

Our approach originates from the 1949 paper:

ITERATES OF FRACTIONAL ORDER

RUFUS ISAACS

1. Introduction. The body of this paper is a complete answer to the following question:

Let E be any space whatever. $g(x)$ is a function[1] mapping E into E. When does there exist a function $f(x)$, of the same type, such that

$$f(f(x)) = g(x) \qquad (x \in E)? \tag{1}$$

This problem typifies the general one of iteration. Let $g^k(x)$ be the kth order iterate of g [i.e. $g^0(x) = x$, $g^{k+1}(x) = g(g^k(x))$]. The iteration problem is that of attaching a consistent meaning to this expression for fractional k (in the sense of preserving the additive law of exponents). An f satisfying (1) is thus $g^{1/2}(x)$. By ideas similar to those discussed herein, we can find the most general $g^{1/m}$ and then by iterating it, the most general iterate of any rational order. Without introducing continuity, this is as far as it is possible to go. We confine ourselves to the case of $k = 1/2$ to avoid oppressive detail; the generalization to $k = 1/m$ is indicated later.

The iteration problem has received attention for many years, alone or as part of another topic (functional equations, fractional derivatives, the tri-operational algebra of Menger [1], etc.). Some of these applications require subsidiary conditions on the functions (continuity, differentiability, etc.). We deal with the general problem without such side conditions; thus our work might be called combinatorial. The problem with a side condition such as continuity appears highly interesting.

In all the literature we have encountered, the general problem is approached in but one way—through the Abel function. The idea here is to ascertain a numerically valued function ϕ on E satisfying

$$\phi(g(x)) = \phi(x) + 1.$$

Then iterates of all orders are obtained at once by

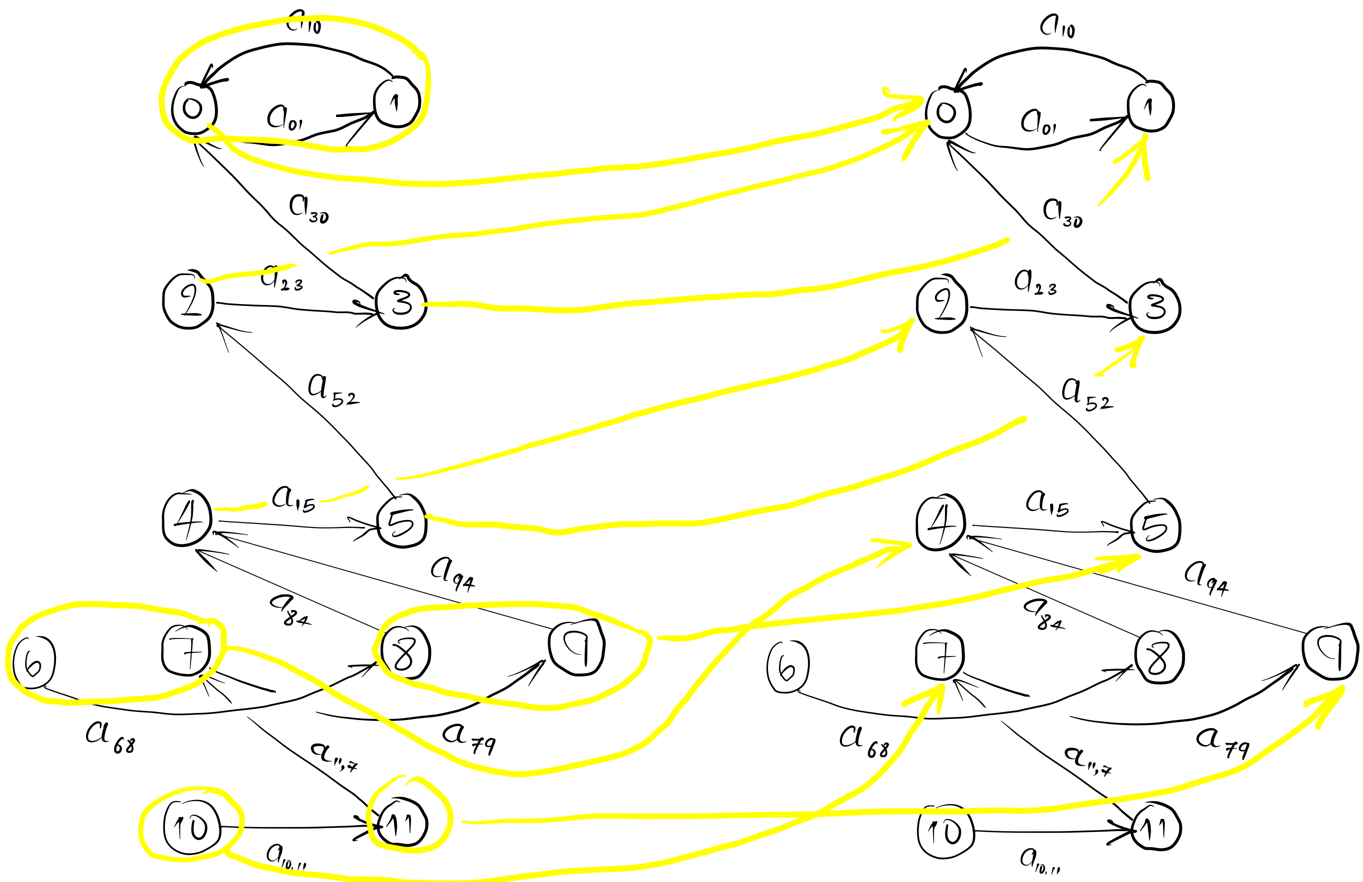
$$g^k(x) = \phi^{-1}(\phi(x) + k).$$

We show later that in a widespread class of cases, a ϕ does not exist. Even when it does, its inverse may not exist. Yet iterates of some or all fractional orders may exist. The non-existence of ϕ may hold even when we have continuity with respect to both x and k, as we shall show below.

Received April 12, 1949.

[1] If g is not defined for all of E, it suffices that our later criterion hold for some extension of g which is. If the range and domain of g are distinct we can thus take E to be their union.

Illustration of Isaac's Necassary & Sufficient Condition



First copy of the Graph of f

Second copy of the Graph of f

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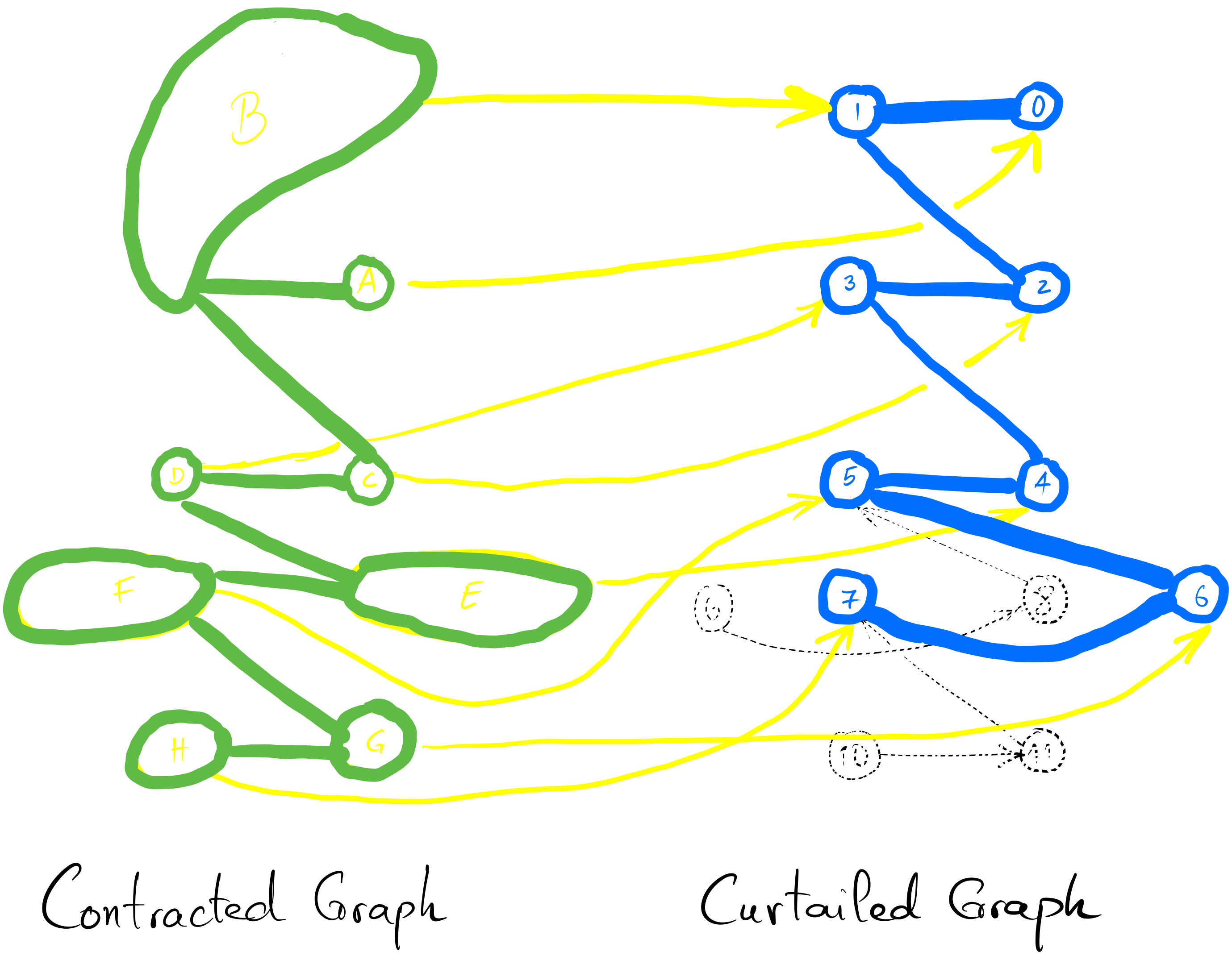


Isaac's Graph Isomorphism

Functional Directed Graphs

Let $\mathbb{Z}_n := [0, n) \cap \mathbb{Z}_n$

$f : \mathbb{Z}_n \to \mathbb{Z}_n$ lies

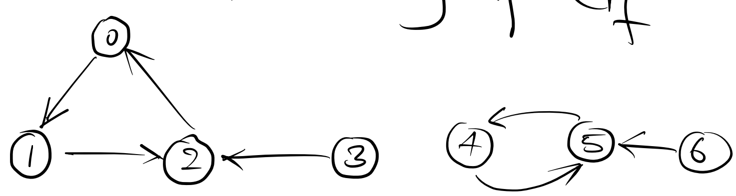
in the transformation monoid $\mathbb{Z}_n^{\mathbb{Z}_n}$.

To such a function we associate a graph

$$G_f := \left(V(G_f) := \mathbb{Z}_n, E(G_f) := \{ (i, f(i)) : i \in \mathbb{Z}_n \} \right)$$

<u>Example</u>: Take $n = 7$, $f \in \mathbb{Z}_7^{\mathbb{Z}_7}$ such that $f(0) = 1$, $f(1) = 2$, $f(2) = 0$, $f(3) = 2$, $f(4) = 5$, $f(5) = 4$, $f(6) = 5$.

Drawing of G_f



τ–Induced Edge Labels

Setup:

Let $\tau : \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{Z}_n$ i.e. $\tau \in \mathbb{Z}_n^{\mathbb{Z}_n \times \mathbb{Z}_n}$

The labeling of the digraph G_f is τ–Zen if

$$\mathbb{Z}_n = \left\{ \tau\left(i, f\left(i\right)\right) \, : \, i \in \mathbb{Z}_n \right\}.$$

The digraph G_f is τ–Zen if

$$n = \max_{\sigma \in \mathrm{S}_n} \left| \left\{ \tau\left(i, f\left(i\right)\right) \, : \, i \in \mathbb{Z}_n \right\} \right|.$$

τ–Induced Edge Labels

Some choices for the function $\tau \in \mathbb{Z}_n^{\mathbb{Z}_n \times \mathbb{Z}_n}$:

- The *bijectivity* choice $\tau(u, v) = v$, for all $(u, v) \in \mathbb{Z}_n \times \mathbb{Z}_n$.
- The *harmonious* choice $\tau(u, v) = u + v \mod n$, for all $(u, v) \in \mathbb{Z}_n \times \mathbb{Z}_n$.
- The *graceful* choice $\tau(u, v) = |v - u|$, for all $(u, v) \in \mathbb{Z}_n \times \mathbb{Z}_n$.
- The *graceful dual* choice $\tau_f(u, v) = \left(u + \sum_{i \in f^{-1}(\{u\}) \backslash \{u\}} i\right) \mod n$, for all $(u, v) \in \mathbb{Z}_n \times \mathbb{Z}_n$.

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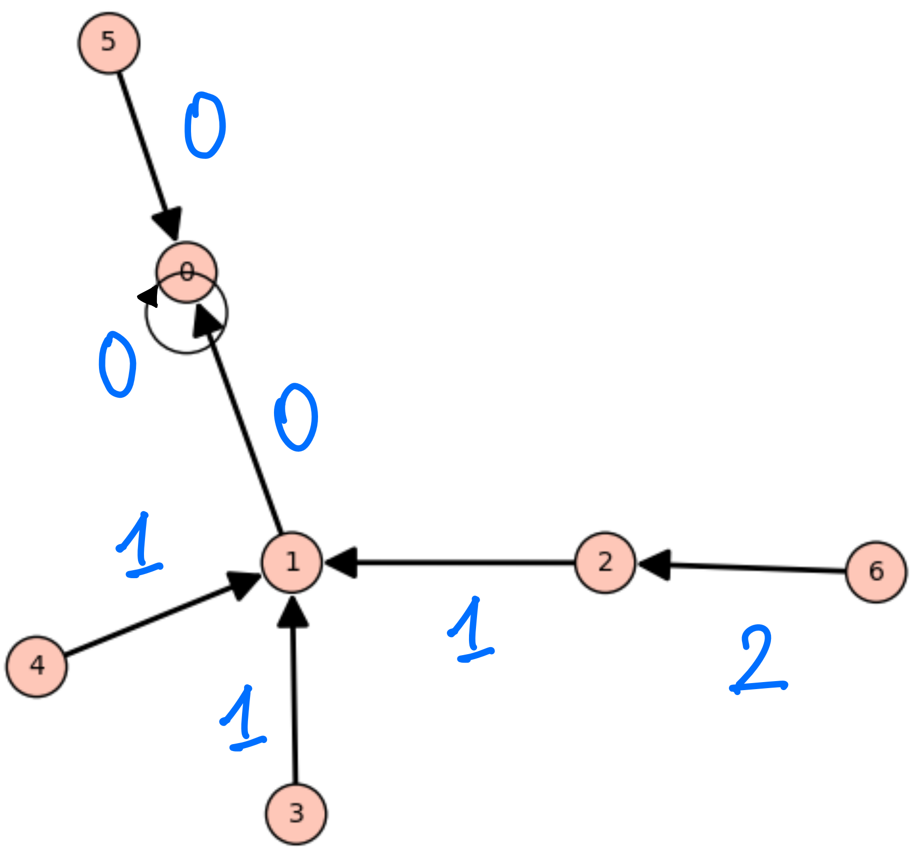
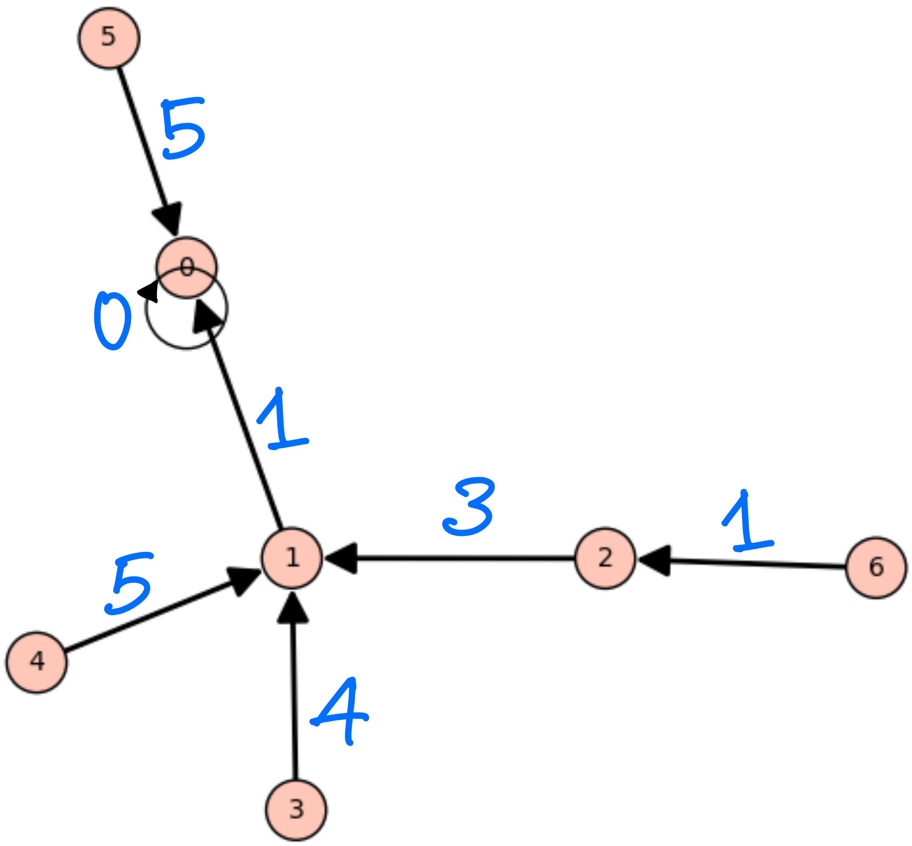
Example:

$$\tau(u,v) = v, \ \forall (u,v) \in \mathbb{Z}_n \times \mathbb{Z}_n$$



Example:

$$\tau(u,v) = u + v \text{ modulo } n, \quad \forall (u,v) \in \mathbb{Z}_n \times \mathbb{Z}_n$$



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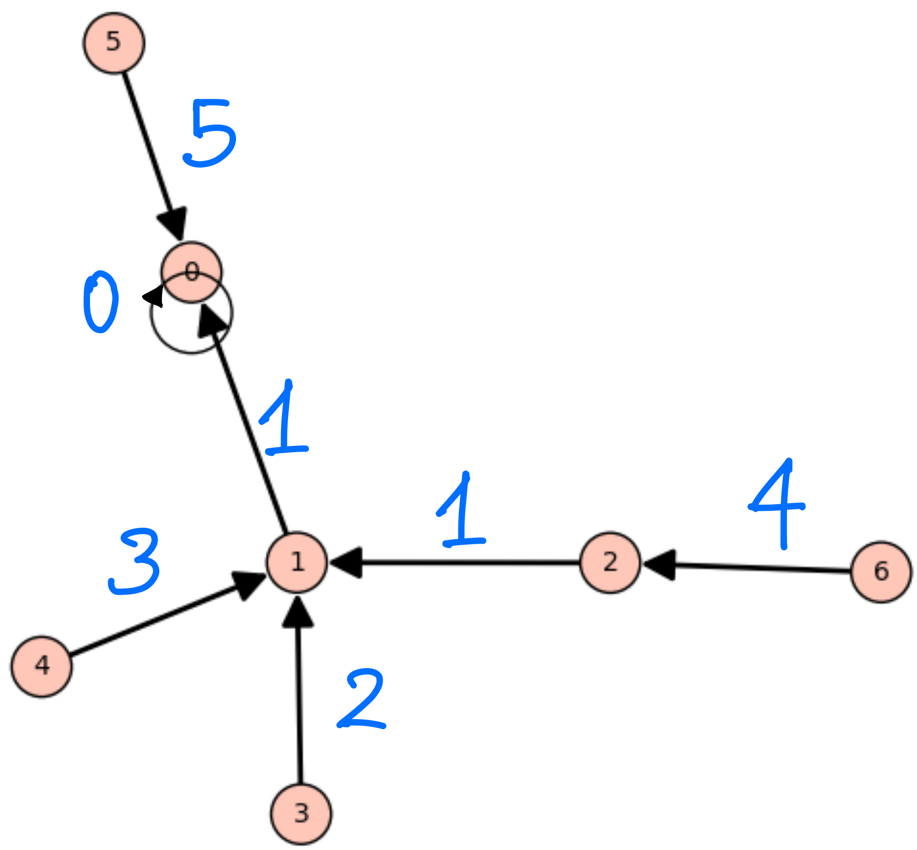

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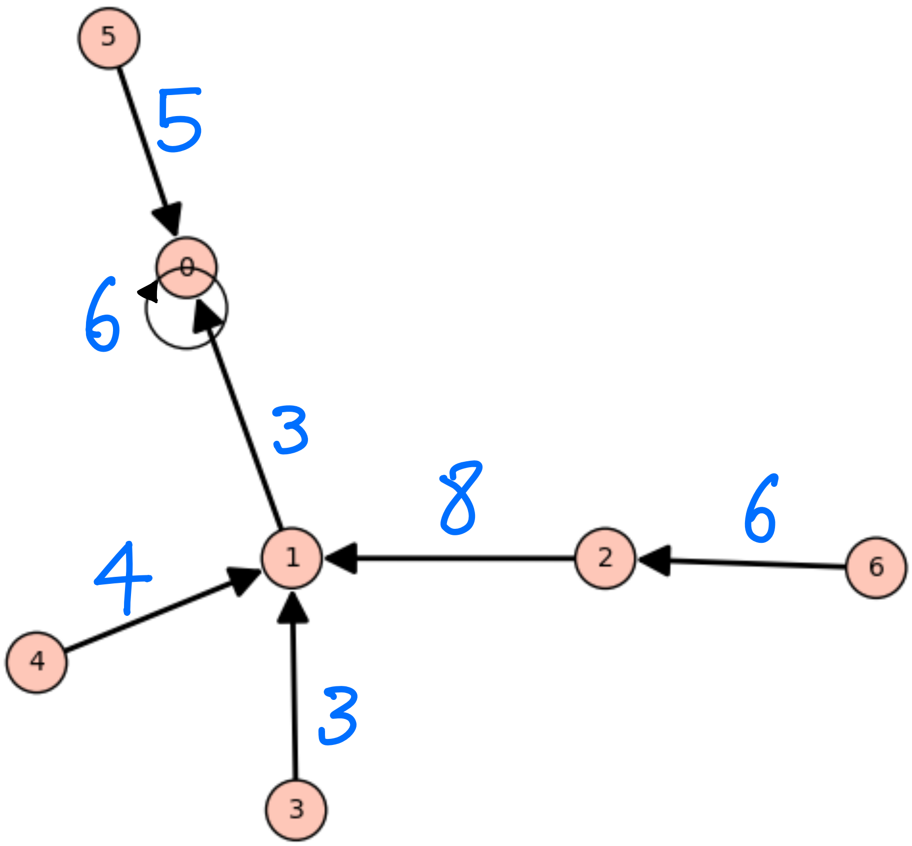

Fundamental Problems

Fundamental Problem 1: Given a family of subset $\mathcal{F}_n \subseteq \mathbb{Z}_n^{\mathbb{Z}_n}$ parametrized by n, determine a necessary and sufficient condition on a given $\tau \in \mathbb{Z}_n^{\mathbb{Z}_n \times \mathbb{Z}_n}$ to ensure that the graphs of every function $f \in \mathcal{F}_n$ is τ–Zen.

Fundamental Problem 2: Given a family of subset $\mathcal{T}_n \subseteq \mathbb{Z}_n^{\mathbb{Z}_n \times \mathbb{Z}_n}$ parametrized by n, determine a necessary and sufficient condition on a given $f \in \mathbb{Z}_n^{\mathbb{Z}_n}$ to ensure that every choice of $\tau \in \mathcal{T}_n$ the graph f is τ–Zen.

Listing τ–Zen Labelings

Let $\text{ZeL}(G_f)$ denote the set of digraphs isomorphic to G_f and whose labeling is τ–Zen. The listing of functional digraph whose labeling is τ–Zen

$$\left(\frac{\partial^n}{\prod_{k\in\mathbb{Z}_n}\partial x_k}\right)\left(\prod_{i\in\mathbb{Z}_n}\sum_{j\in\mathbb{Z}_n} a_{i,j}\, x_{\tau(i,j)}\right) = \left(\sum_{\substack{f\,\in\,\mathbb{Z}_n^{\mathbb{Z}_n} \\ G_f\,\in\,\text{ZeL}\,(G_f)}} \prod_{i\in\mathbb{Z}_n} a_{i,f(i)}\right).$$

It follows that the listing/enumeration of graphs whose labeling is τ–Zen amounts to computing a Permanent.

By Symmetrization :

$$F_{\tau}\left(\mathbf{A}\right)=\left(\frac{\partial^{n}}{\prod_{k\in\mathbb{Z}_{n}}\partial x_{k}}\right)\left(\sum_{\sigma\in S_{n}}\prod_{i\in\mathbb{Z}_{n}}\sum_{j\in\mathbb{Z}_{n}}a_{\sigma(i),\sigma(j)}x_{\tau(i,j)}\right)=$$

$$\sum_{f\in\mathbb{Z}_{n}^{\mathbb{Z}_{n}}}\left|\text{Aut}\left(G_{f}\right)\right|\left|\text{ZeL}\left(G_{f}\right)\right|\prod_{i\in\mathbb{Z}_{n}}a_{i,f(i)}.$$

Main results

Let $\mathbf{A}, \mathbf{X}_\tau$ denote symbolic $n \times n$ matrices with entries given by

$$\mathbf{A}[i,j] = a_{i,j}, \quad \mathbf{X}_\tau[i,j] = x_{\tau(i,j)}, \quad \forall\,(i,j) \in \mathbb{Z}_n \times \mathbb{Z}_n$$

For an instance of the first fundamental problem take the family of functional trees

$$T_n = \left\{ f \in \mathbb{Z}_n^{\mathbb{Z}_n} : 1 = \left| f^{(n-1)}(\mathbb{Z}_n) \right| \right\}.$$

By Tutte's Directed Matrix Tree Theorem

$$t_n(\mathbf{A}) = \sum_{i \in \mathbb{Z}_n} a_{i,i} \det\left\{ \left(\operatorname{diag}\left(\mathbf{A}\mathbf{1}_{n\times 1}\right) - \mathbf{A}\right)_{i,i} \right\} = \sum_{f \in T_n} \prod_{i \in \mathbb{Z}_n} a_{i,f(i)}.$$

Main results

A necessary and sufficient condition

on τ which ensures that every member of T_n is τ–Zen is that

$$x^{n^{n-1}} + \sum_{0<k\leq n^{n-1}} (-1)^k \, x^{n^{n-1}-k} \sum_{\substack{m_1+2m_2+\cdots+km_k=n^{n-1} \\ m_1\geq 0,\ldots,m_k\geq 0}} \prod_{0<i\leq n^{n-1}} \frac{\left(-t_n\left(\mathbf{A}^{\circ^k}\right)\right)^{m_i}}{m_i! \, i^{m_i}},$$

divides the polynomial

$$x^{\eta} + \sum_{0<k\leq \eta} (-1)^k \, x^{\eta-k} \sum_{\substack{m_1+2m_2+\cdots+km_k=\eta \\ m_1\geq 0,\ldots,m_k\geq 0}} \prod_{0<i\leq \eta} \frac{\left(-F_\tau\left(\mathbf{A}^{\circ^k}\right)\right)^{m_i}}{m_i! \, i^{m_i}},$$

where $\eta = F_\tau\left(\mathbf{1}_{n\times n}\right)$.

Main results

Strengthening the Kotzig–Ringel–Rosa conjecture

Take $\tau \in \mathbb{Z}_n^{\mathbb{Z}_n \times \mathbb{Z}_n}$ such that

$$\tau(u,v) = |v-u|, \ \forall \ (u,v) \in \mathbb{Z}_n \times \mathbb{Z}_n.$$

In this setting the entries of $\mathbf{X}_\tau \in (\mathbb{Q}[x_0, \cdots, x_{n-1}])^{n \times n}$ are such that

$$\mathbf{X}_\tau[u,v] = x_{|v-u|}, \ \forall \ (u,v) \in \mathbb{Z}_n \times \mathbb{Z}_n.$$

The strong form of the KRRC asserts that the only solutions $g \in \mathbb{Z}_n^{\mathbb{Z}_n}$ are identically constant in

$$0 = \sum_{\gamma \in \mathrm{S}_n} \sum_{f \in T_n} \prod_{\sigma \in \mathrm{S}_n} \left(\prod_{i \in \mathbb{Z}_n} \mathbf{X}_\tau \left[i, \gamma g \gamma^{-1}(i)\right] - \prod_{i \in \mathbb{Z}_n} \mathbf{X}_\tau \left[i, \sigma f \sigma^{-1}(i)\right] \right)$$

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Composition Lemma

Theorem: Let $T_n = \left\{ g \in \mathbb{Z}_n^{\mathbb{Z}_n} : 1 = \left| g^{(n-1)}\left(\mathbb{Z}_n\right) \right| \right\}$, given $\tau \in \mathbb{Z}_n^{\mathbb{Z}_n \times \mathbb{Z}_n}$ such that identically constant functions in T_n are τ–Zen and

$$\max_{\sigma \in \text{S}_n} \left| \left\{ \tau\left(i, \sigma f^{(2)} \sigma^{-1}(i)\right) : i \in \mathbb{Z}_n \right\} \right| \leq \max_{\sigma \in \text{S}_n} \left| \left\{ \tau\left(i, \sigma f \sigma^{-1}(i)\right) : i \in \mathbb{Z}_n \right\} \right|.$$

then the graph of any member of T_n is τ–Zen.

Proof: Observe $f \in T_n \implies f^{(2)} \in T_n$. For any $f \in T_n$ consider the $\lceil \lg(n-1) \rceil$–sequence

$$\left(f = f^{\left(2^0\right)}, f^{\left(2^1\right)}, \cdots, f^{\left(2^{\lceil \lg(n-1) \rceil - 1}\right)}, f^{\left(2^{\lceil \lg(n-1) \rceil}\right)} = \text{constant} \right).$$

Constants in T_n being τ–Zen and repeatedly invoking the inequality yields that members of the sequence are all τ–Zen.