# The Smith normal form of Hankel matrices of sequences defined by exponential generating functions

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A matrix of integers is in *Smith normal form* if it is of the form

$$\mathrm{diag}(\lambda_1, \lambda_2, \dots) = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \lambda_r & \\ 0 & 0 & \cdots & & \ddots \end{bmatrix}.$$

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Combinatorial aspects of Smith normal forms were studied by Stanley and others.

For example, the Smith normal form of the following matrix, a Hankel matrix whose entries are Bell numbers,

$$\begin{bmatrix} 1 & 1 & 2 & 5 & 15 \\ 1 & 2 & 5 & 15 & 52 \\ 2 & 5 & 15 & 52 & 203 \\ 5 & 15 & 52 & 203 & 877 \end{bmatrix}$$

is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \end{bmatrix}.$$

A useful characterization of the Smith normal form of $M$ is that if the diagonal entries of the Smith normal form of $M$ are $\lambda_1, \ldots, \lambda_n$ then for each $i$, the product $\lambda_1\lambda_2\cdots\lambda_i$ is the greatest common divisor of all determinants of $i \times i$ minors of $M$.

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So Smith normal forms of Hankel matrices of sequences defined by exponential generating functions are nontrivial and they are (empirically) interesting.

Let's define the *Smith sequence* of an exponential generating function

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$$

to be the sequence $\lambda_2/\lambda_1, \lambda_3/\lambda_2, \ldots$ where $\mathrm{diag}(\lambda_1, \lambda_2, \ldots)$ is the Smith normal form of the infinite Hankel matrix for $\mathbf{a}$.

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For example the Smith sequence for $e^{e^x-1}$ (Bell numbers) is $1, 2, 3, 4, 5, 6, \ldots$. We can prove this because the Hankel matrix has a nice LDU decomposition. (Related to orthogonal polynomials and continued fractions—studied by Stanton and Miller.) My other examples are all just empirical.

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The Smith sequence for $e^{1-\sqrt{1-2x}}$ is also $1, 2, 3, 4, 5, 6, \ldots$.

The Smith sequence for $e^{\sinh x}$ is $1, 1, 9, 4, 25, 2, 49, 1, 81, 2, 121, 36, 169, 1, 25, 16, 289, 18, 361, \ldots$ The entries are all squares or twice squares.

The Smith sequence for

$$\frac{-2\log(1-x)}{2-x} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \sum_{i=0}^{n} i!\,(n-i)!$$

is

$1, 1, 1, 16, 1, 27, 1, 256, 9, 125, 1, 144, 1, 343, 225, 4096, 1, 243, \ldots$

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The 1s occur exactly in the prime positions! (And position 1.)

The entries are almost all perfect powers.

The Smith sequence for both $\cosh(e^x - 1)$ and $\sinh(e^x - 1)$ is

$$1, 2, 1, 4, 1, 6, 1, 8, 1, 10, 1, 12, 1, 14, 1, 16, 1, 18, \ldots$$

The Smith sequence for $\cosh(2\sinh(x/2))$ is

$$1, 1, 1, 4, 1, 9, 1, 4, 1, 25, 1, 4, 1, 49, 1, 16, 1, 81, 1, 8, 1, 121, 1, 9, 1, \ldots$$

The Smith sequence for $3x/(1 + e^x + e^{2x})$ is

$$1, 6, 1, 12, 1, 2916, 1, 64, 1, 5000, 1, 2916, 1, 14406, 1, 15360, 1, \ldots$$

The Smith sequence for $e^{xe^x}$ is

$$1, 1, 1, 4, 1, 2, 1, 1, 9, 2, 1, 4, 1, 1, 1, 16, 3, 2, 3, 1, 1, 4, 1, 1, 75, \ldots$$