My Mathematical Memories and More with Adriano

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Garsia Seminar

March 6th, 2025

Celebrating the life and mathematics of Adriano and the

many lessons I learned from him.


MAR 25 2005

Theorem 3 (Garsia ? 2004) *For an elliptic curve, we can write N_k as a polynomial in terms of N_1 and q such that*

$$N_k = \sum_{i=1}^{k} (-1)^{i-1} P_{k,i}(q) N_1^i$$

where each $P_{k,i}$ is a polynomial in q with positive integer coefficients.

This can be proven using the fact that

$$N_k = 1 + q^k - \alpha_1^k - \alpha_2^k$$

and this leads to a recursion for $\alpha_1^k + \alpha_2^k$ in terms of

$$\begin{aligned} \alpha_1 + \alpha_2 &= 1 + q - N_1 \quad \text{and} \\ \alpha_1 \alpha_2 &= q. \end{aligned}$$

We can prove positivity by induction.

$$
\begin{aligned}
N_2 &= (2+2q)N_1 - N_1^2 \\
N_3 &= (3+3q+3q^2)N_1 - (3+3q)N_1^2 + N_1^3 \\
N_4 &= (4+4q+4q^2+4q^3)N_1 - (6+8q+6q^2)N_1^2 + (4+4q)N_1^3 - N_1^4 \\
N_5 &= (5+5q+5q^2+5q^3+5q^4)N_1 - (10+15q+15q^2+10q^3)N_1^2 \\
&+ (10+15q+10q^2)N_1^3 - (5+5q)N_1^4 + N_1^5
\end{aligned}
$$

Question 1 *What is a combinatorial interpretation of these expressions, i.e. of the $P_{k,i}$'s?*

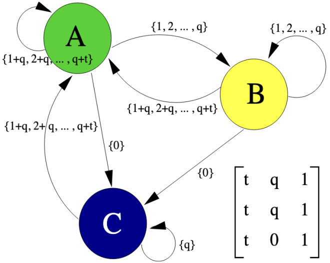

Figure 6.2: Deterministic finite automaton M_G.

Regular cyclic languages such as $\mathcal{L}(q,t)$ were studied in [BR90], and we can even define a zeta function for them. The zeta function of a cyclic language L is defined as

$$\zeta(L,T) = \exp\left(\sum_{k=1}^{\infty} \mathcal{W}_k \frac{T^k}{k}\right)$$

where $\mathcal{W}_k$ is the number of words of length k. Alternatively, this can be written as

$$\zeta(L,T) \;=\; \exp\left(\sum_{\text{allowed closed paths } P} (\text{\# words admissible by path } P)\;\; T^k\right).$$

Theorem 6.20 (Berstel and Reutenauer)**.** *The zeta function of a cyclic and regular language is rational.*

1 Quasiinvariants with a nice form

We begin by defining the following elements of the group algebra of S_3:

$$[S_3] = \frac{1}{6}\sum_{\sigma\in S_3}\sigma, \qquad [S_3]' = \frac{1}{6}\sum_{\sigma\in S_3}\mathrm{sgn}(\sigma)\sigma$$

$$\pi_1 = \frac{1}{3}(1+s_{23})(1-s_{12}), \qquad \pi_2 = \frac{1}{3}(1+s_{12})(1-s_{23})$$

These defined, the following identities are easily verified:

$$(\pi_1)^2 = \pi_1, (\pi_2)^2 = \pi_2 \tag{4}$$

$$[S_3]'\pi_1 = \pi_1\pi_2 = \pi_2\pi_1 = 0 \tag{5}$$

$$[S_3] + \pi_1 + \pi_2 + [S_3]' = 1 \tag{6}$$

$$s_{23}\pi_1 = \pi_1 \tag{7}$$

$$\pi_2 s_{12}\pi_1 = -s_{13}\pi_1 \tag{8}$$

We now show that there exist quasiinvariants satisfying certain symmetry and independence conditions.

Lemma 1. *For all $m \geq 0$, there exist non-symmetric m-quasiinvariants A_1, A_2 of degrees $3m+1, 3m+2$, respectively, such that $s_{23}(A_i) = A_i$ and in the quotient $QI_m/\langle(e_1, e_2, e_3)\rangle$, the image of A_i and $s_{12}(A_i)$ are linearly independent. Further all four of these will be independent of $\Delta^{2m+1}(x)$.*

Theorem 1. *The vector space of quasiinvariants has the following direct sum decomposition:*

$$\mathbf{QI}_m = \bigoplus_{T \in ST(n)} \left(\gamma_T \mathbf{R} \cap V_T^{2m+1} \mathbf{R}\right)$$

where $ST(n)$ is the set of standard tableaux of size n, γ_T is a projection opertor due to Young (defined in full detail in the next section) and V_T is the polynomial given by the product over the columns of T of the associated "Vandermonde determinants" (this is also defined in detail below). This characterization is proved using completely elementary methods (namely, computations in the group algebra of the symmetric group) in section 4. In section 5 we use this characterization to construct the basis for the $[n-1, 1]$ isotypic component. Precisely, for T a standard Young tableau of shape $[n-1, 1]$ with j the entry in the second row, we set

$$Q_T^{k,m} = \int_{x_1}^{x_j} t^k \prod_{i=1}^{n} (t - x_i)^m dt.$$

With this definition, we have

Theorem 2. *The set*

$$\{Q_T^{0,m}, Q_T^{1,m}, Q_T^{2,m}, \ldots, Q_T^{n-2,m}\}$$

is a basis for $\gamma_T \left(\mathbf{QI}_m / \langle e_1, \ldots, e_n \rangle\right)$.


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Example: Factoring N_6 Completely

$$N_6 = N_1\left(2 + 2q - N_1\right)\left((3 + 3q + 3q^2) - (3 + 3q)N_1 + N_1^2\right)\left((1 - q + q^2) - (1 + q)N_1 + N_1^2\right)$$

$$N_6 = E(\mathbb{F}_{q^6}) = Ker(1 - \pi^6)$$

$$N_2 = E(\mathbb{F}_{q^2}) = Ker(1 - \pi^2) \qquad N_3 = E(\mathbb{F}_{q^3}) = Ker(1 - \pi^3)$$

$$N_1 = E(\mathbb{F}_q) = Ker(1 - \pi)$$

$$Cyc_d(\pi) = \prod_{k|d}(1 - \pi^k)^{\mu(d/k)}$$

$$1 - \pi^6 = (1 - \pi)(1 + \pi)(1 + \pi + \pi^2)(1 - \pi + \pi^2)$$