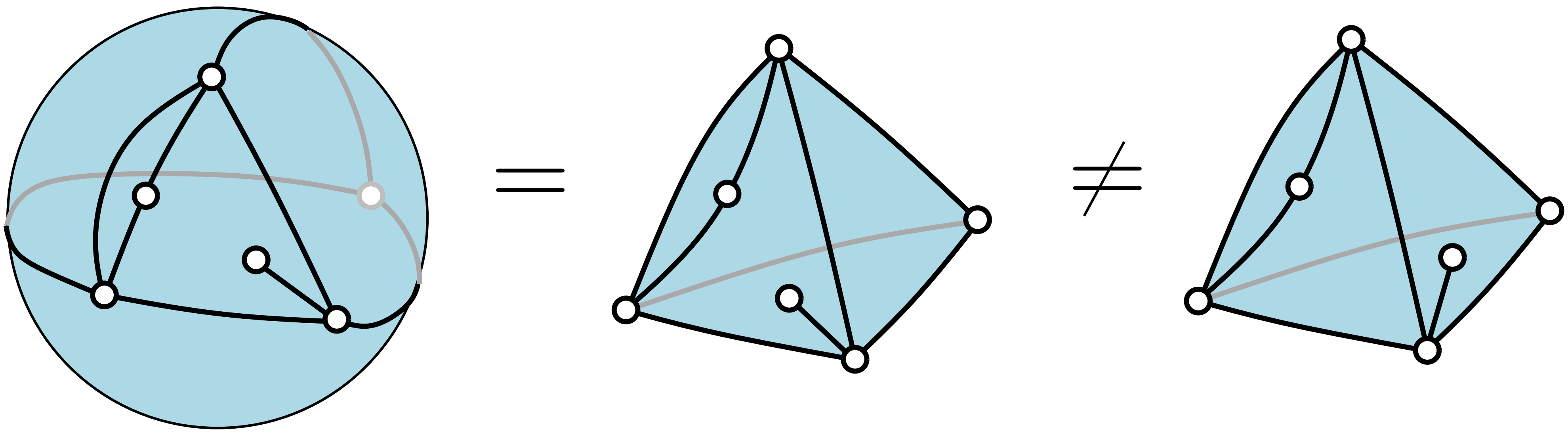
Enumeration of corner polyhedra

Éric Fusy (LIGM, Univ. Gustave Eiffel)
Joint work with Erkan Narmanli and Gilles Schaeffer

Rutgers Experimental Math Seminar, Dec. 12th, 2024

Planar maps

Def. Planar map = connected graph embedded on the sphere



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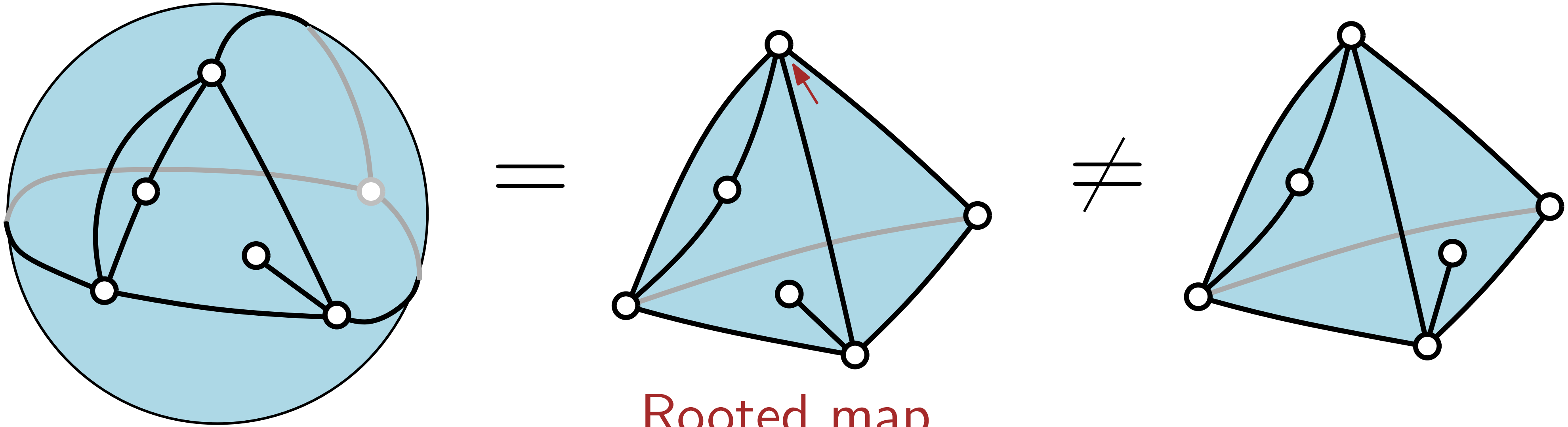

Rooted map
= map with marked corner

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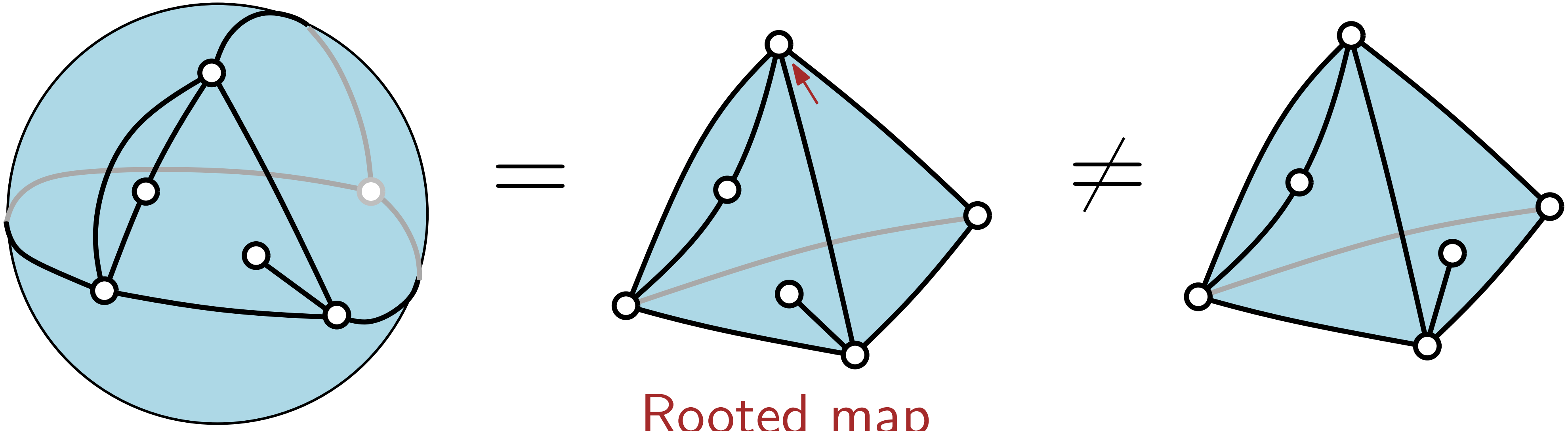
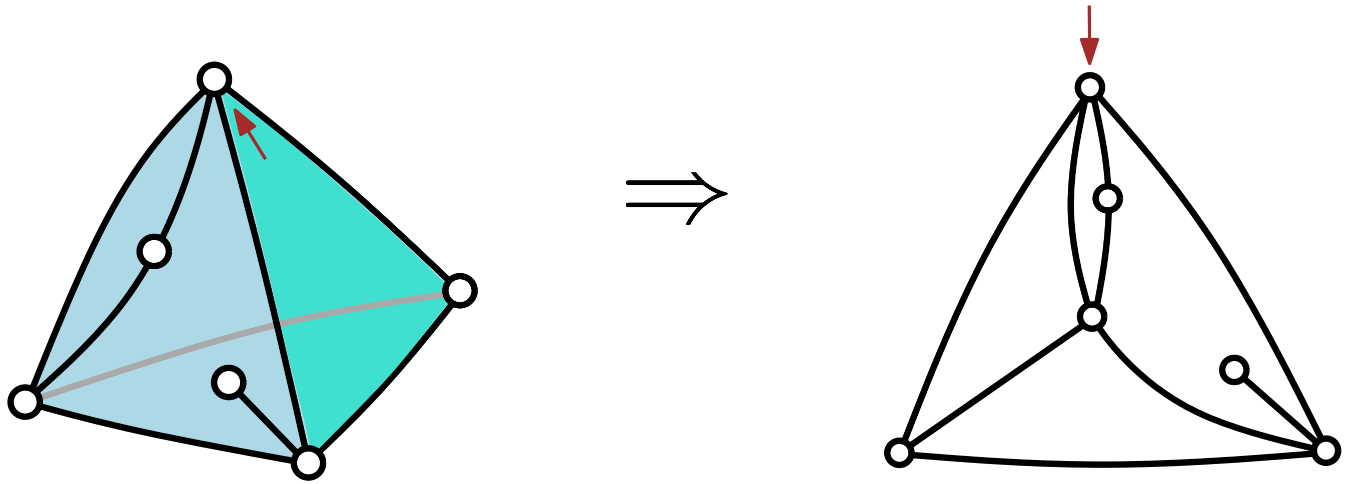


Rooted map
= map with marked corner

Easier to draw in the plane (choosing root-face to be the outer face)



Counting planar maps

- Nice counting formulas [Tutte'62,63]

arbitrary maps n edges

$$\frac{2 \cdot 3^n}{(n+2)(n+1)}\binom{2n}{n}$$

simple quadrangulations n faces

$$\frac{2}{n(n+1)}\binom{3n}{n-1}$$

bipartite maps n edges

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- Counting methods:
  - recursive decomposition & solving functional equations
    [Tutte'63],[Bousquet-Mélou&Jehanne'06],[Eynard'15],...
  - matrix integrals (Feynman diagrams $\approx$ maps)
    ['t Hooft'74], [Brézin et al'78],...
  - bijections (with models of trees that are easy to count)
    [Schaeffer'97],[Bouttier-Di Francesco-Guitter'02],...

Universality for planar maps

- asymptotic behaviour in $c\,\gamma^n\,n^{-5/2}$ (vs correction $n^{-3/2}$ for tree families)

  e.g. $$m_n = \frac{2\cdot 3^n}{(n+2)(n+1)}\binom{2n}{n} \sim \frac{2}{\sqrt{\pi}}12^n\ n^{-5/2}$$

- generating functions typically algebraic

  e.g. $M(t) = \sum_{n\geq 1} m_n t^n$ satisfies

$$27\,t - 2 + \left(54\,t^2 - 18\,t + 1\right) M\left(t\right) + 27\,t^3\left(M\left(t\right)\right)^2 = 0$$

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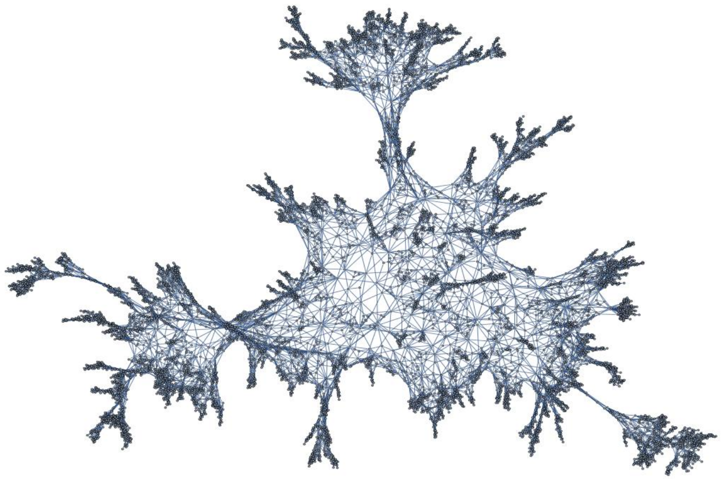
- for a uniform random map of size n (in a given family)

  typical distances behave as $n^{1/4}$ (vs $n^{1/2}$ for tree families)
  [Chassaing,Schaeffer'04]

  universal scaling limit (Brownian map)
  [Le Gall'13, Miermont'13]



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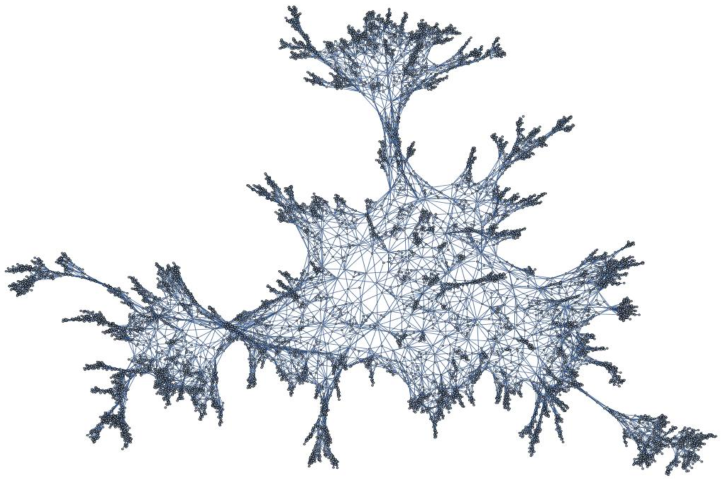

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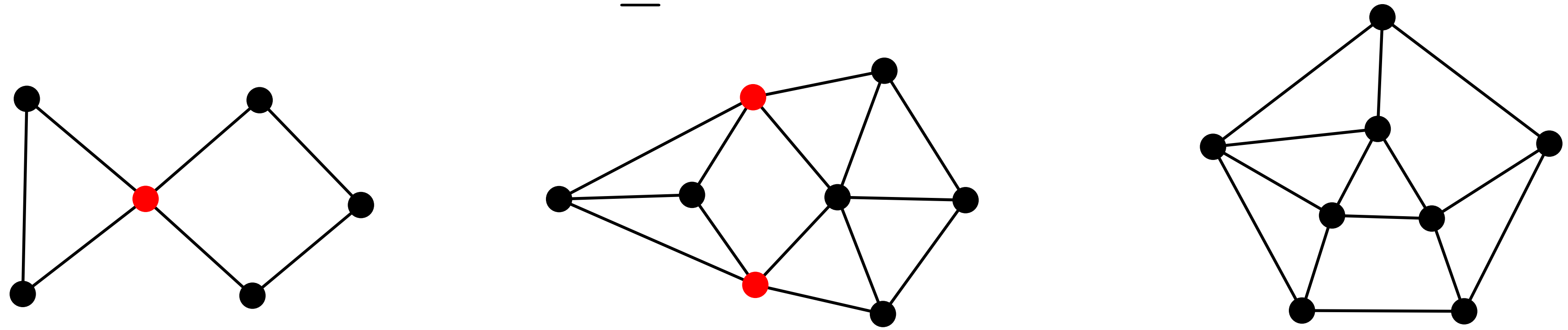
Rk: new asympt. behaviours when considering decorated maps

3-connected maps

k-connected: needs to delete $\geq k$ vertices to disconnect



not 2-connected   not 3-connected   3-connected

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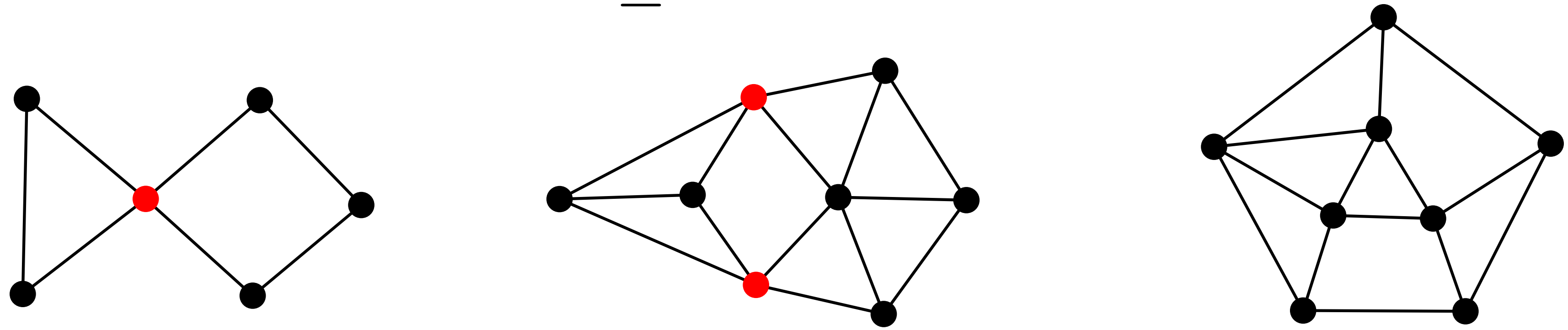

not 2-connected — not 3-connected — 3-connected

Interesting family:

- 3-connected planar maps $\leftrightarrow$ 3-connected planar graphs [Whitney'32]

  → building bricks to count planar graphs (exact & asymptotic)
  [Bender,Gao,Wormald'02], [Giménez,Noy'05]

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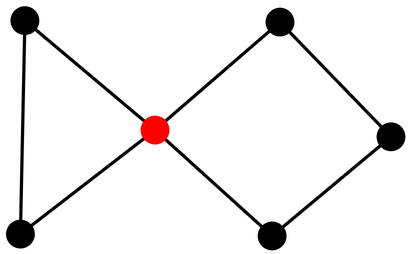

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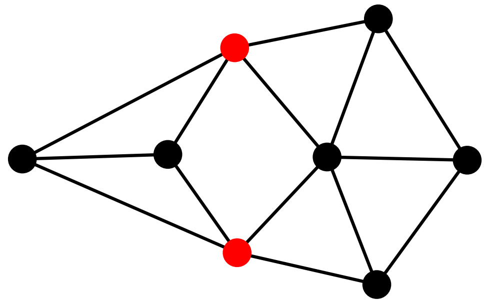

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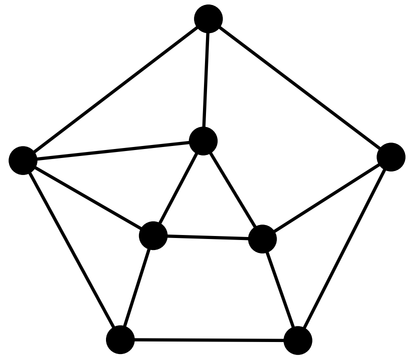

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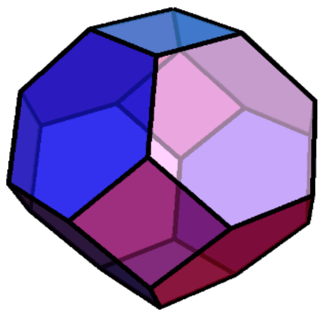

[Steinitz'34]

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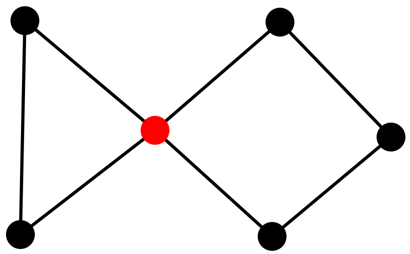

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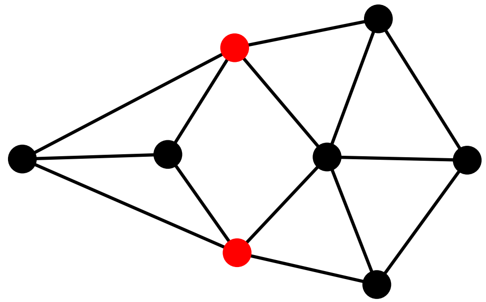

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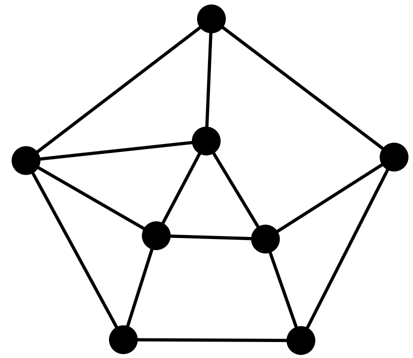

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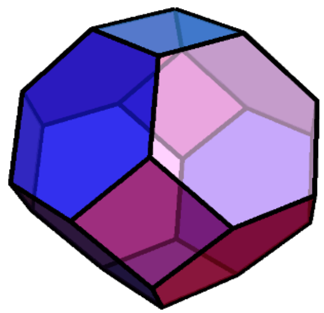


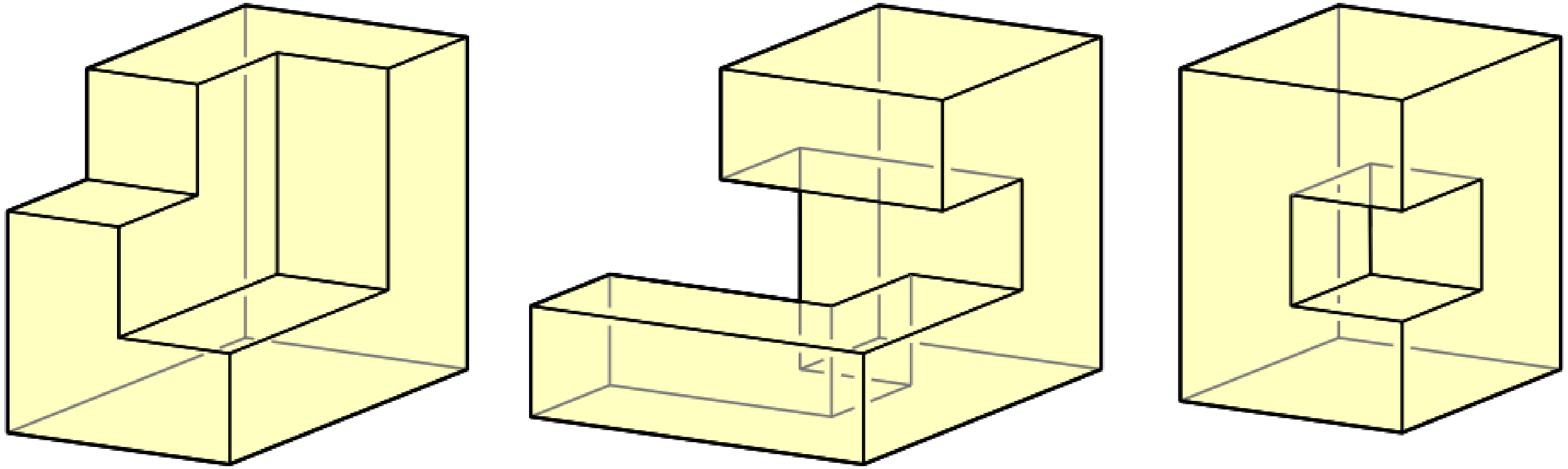
Catalan GF

Enumeration: $M(t) = t^2 \dfrac{1-t}{1+t} - \dfrac{C(t)^2}{(1+2C(t))^3}$

[Mullin, Schellenberg'68]
[F, Poulalhon, Schaeffer'05]

Simple orthogonal polyhedra [Eppstein-Mumford'09]

simple orthogonal polyhedron = 3d polyhedron such that, at each vertex three axis-aligned segments meet



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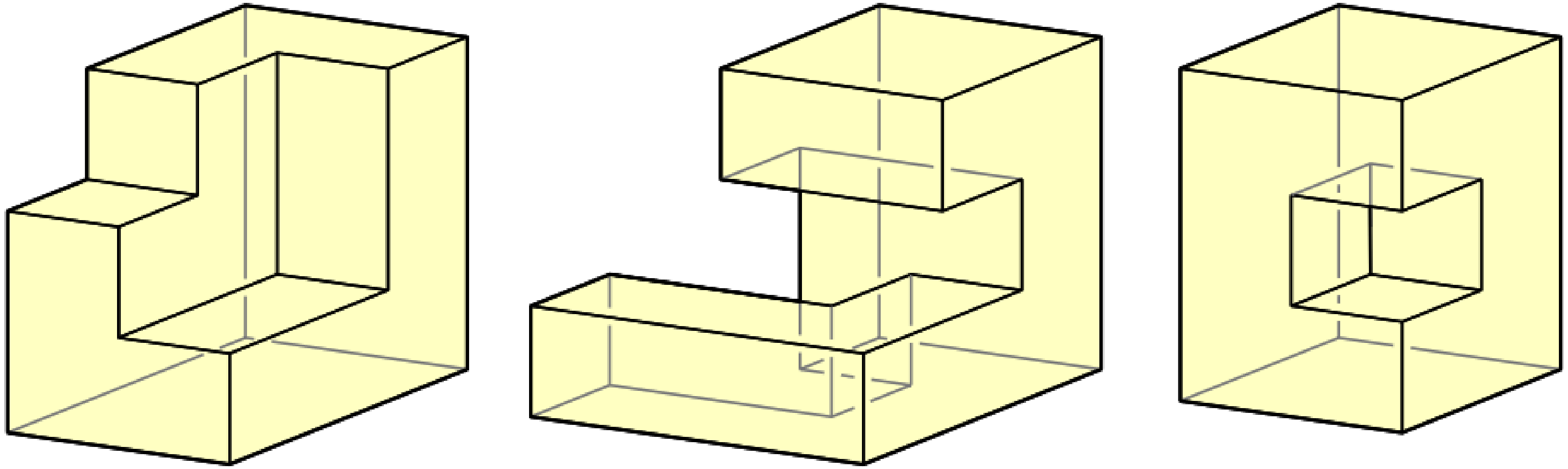

Rk: boundary forms a cubic (and bipartite) map on the sphere

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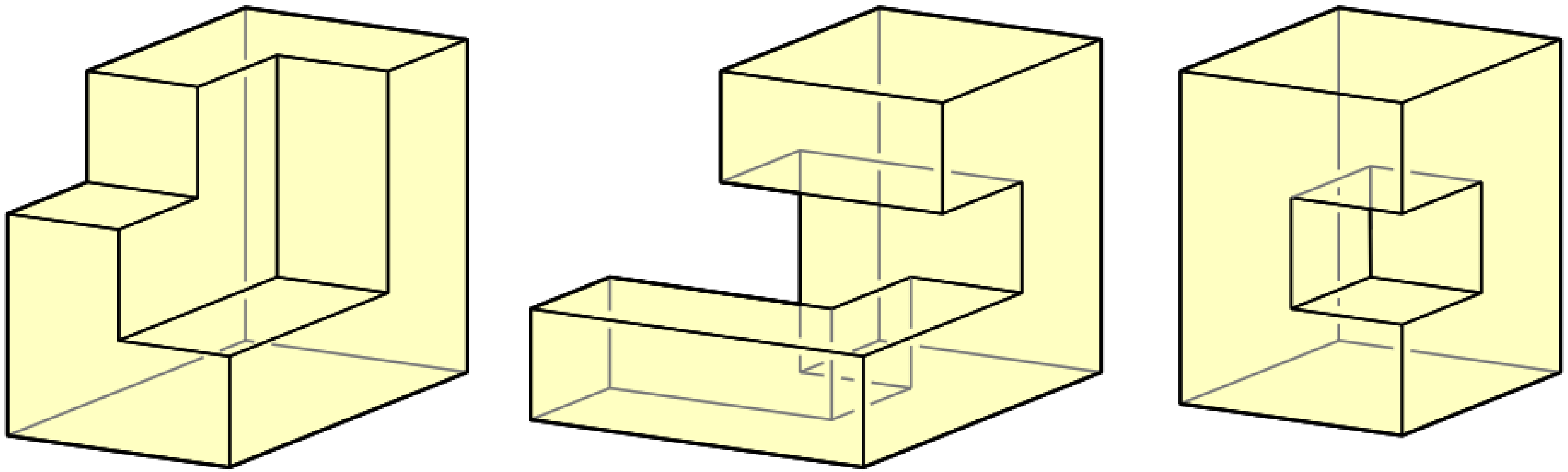

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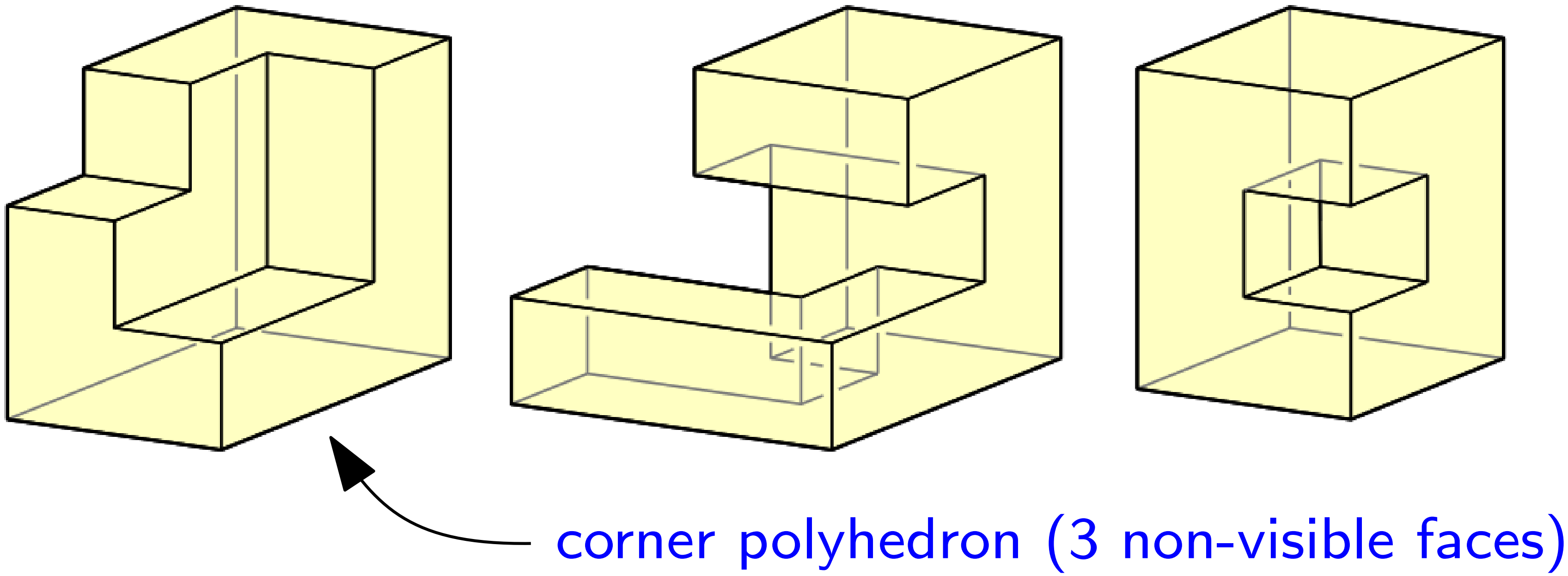

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Characterization of corner polyhedra maps

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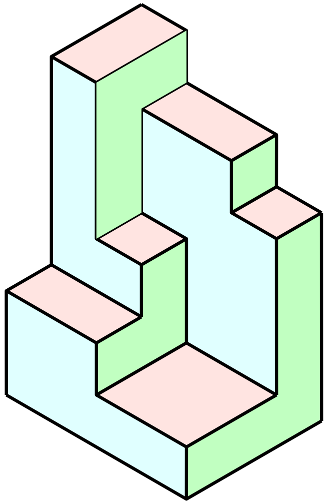

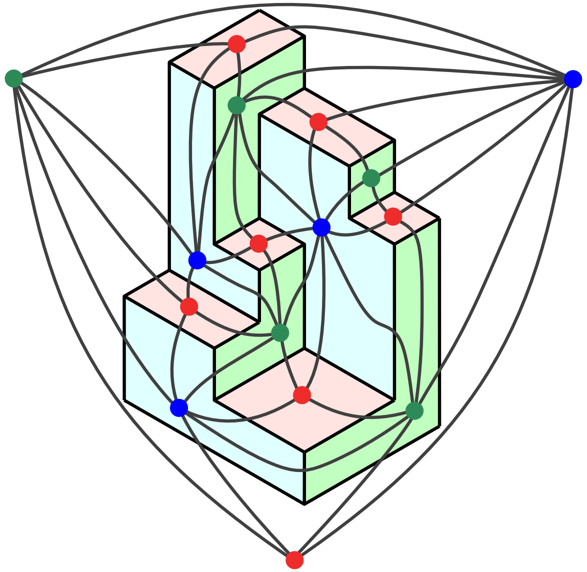

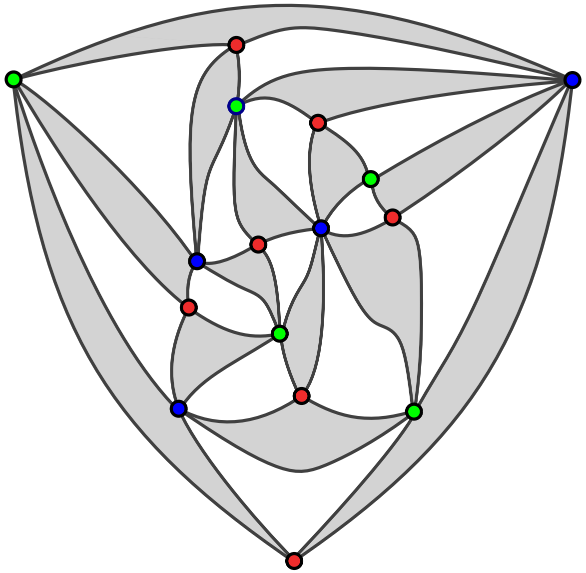

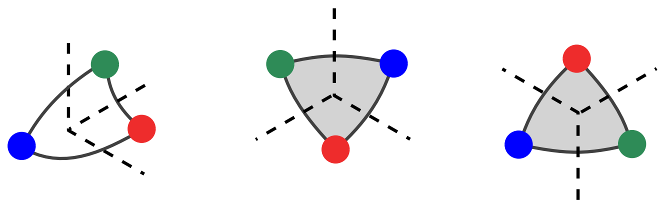

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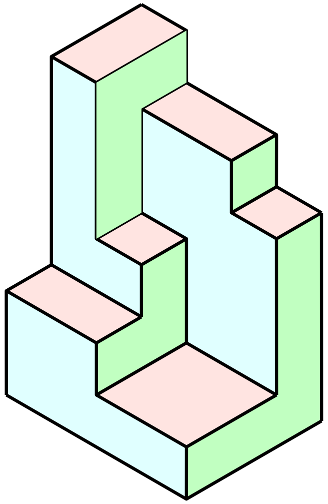
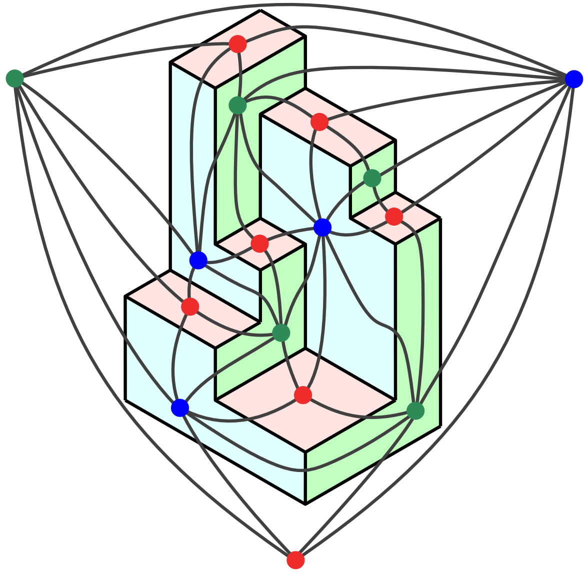
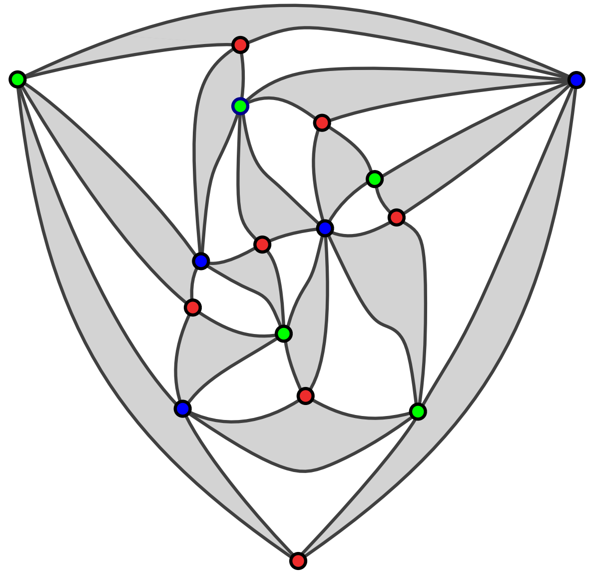
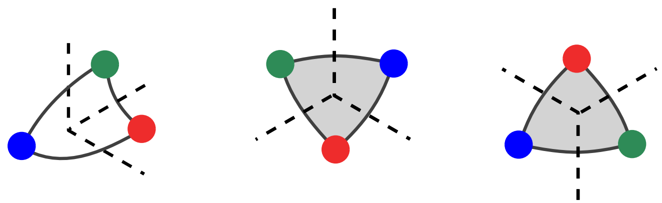
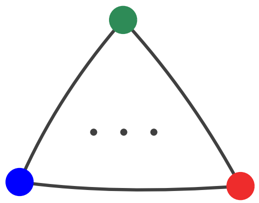








Every  bounds a face

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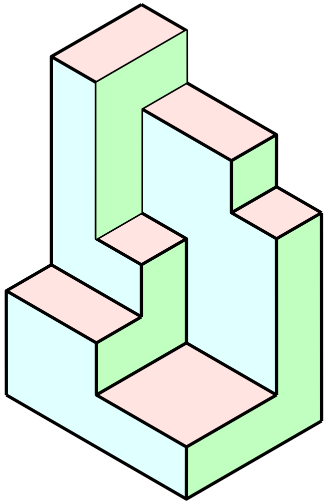
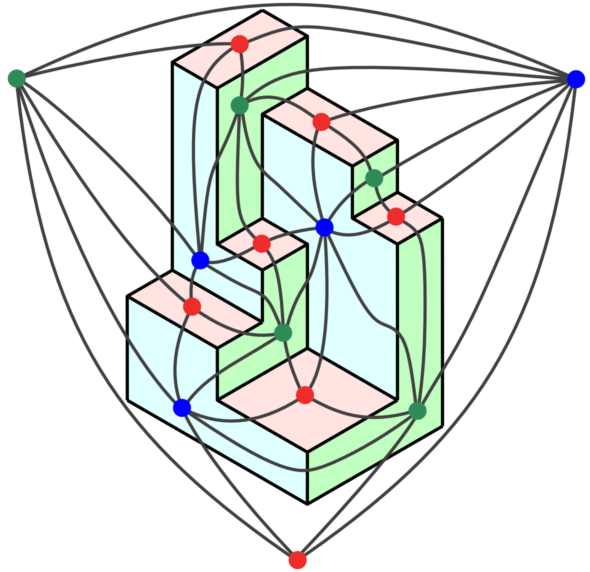
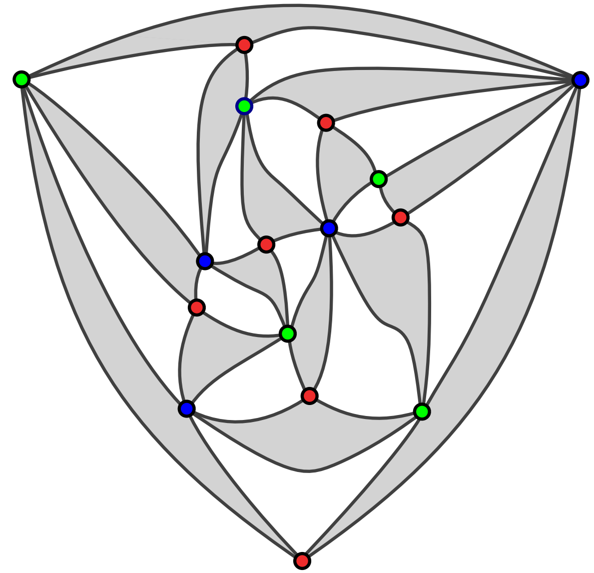
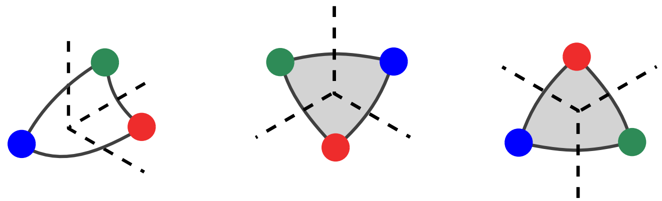
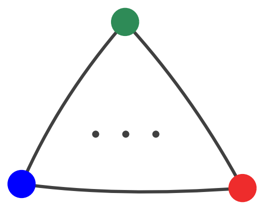








Every  bounds a face

Enumeration of these "corner triangulations": [Dervieux, Poulalhon, Schaeffer'16]

$$C(t) = \sum_n c_n t^n = t^3 + 3t^5 + 4t^6 + 15t^7 + 39t^8 + 120t^9 + \cdots$$

has rational expression in terms of Catalan generating function

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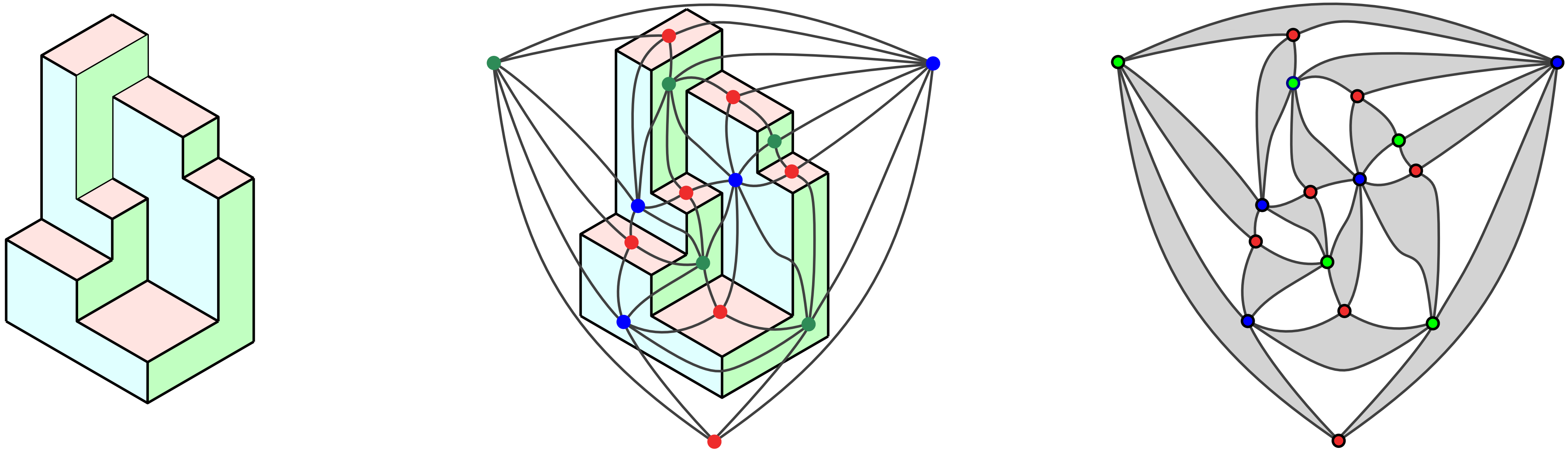
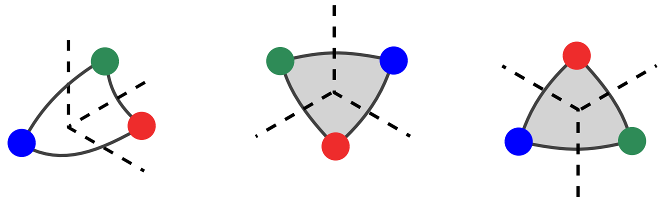
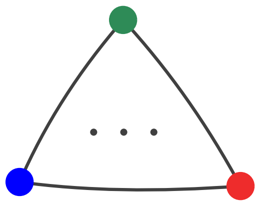




Every  bounds a face

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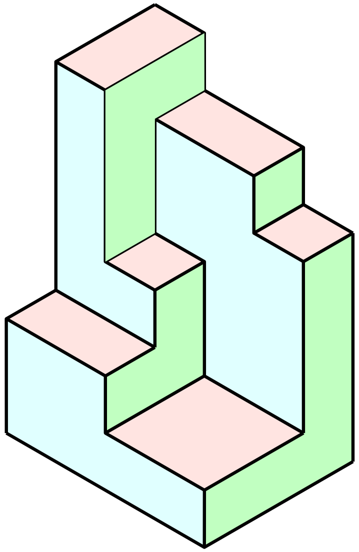
Rk: $C(t)$ = GF of 3-connected maps with root-vertex of degree 3

Enumeration of corner polyhedra



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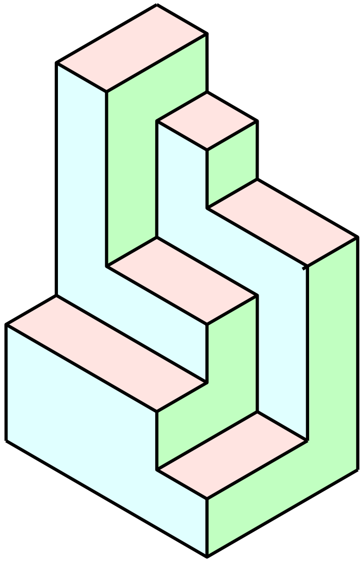

$p_n = \#$ combinatorial types of corner polyhedra of size n

where size $= \#$ flats -3

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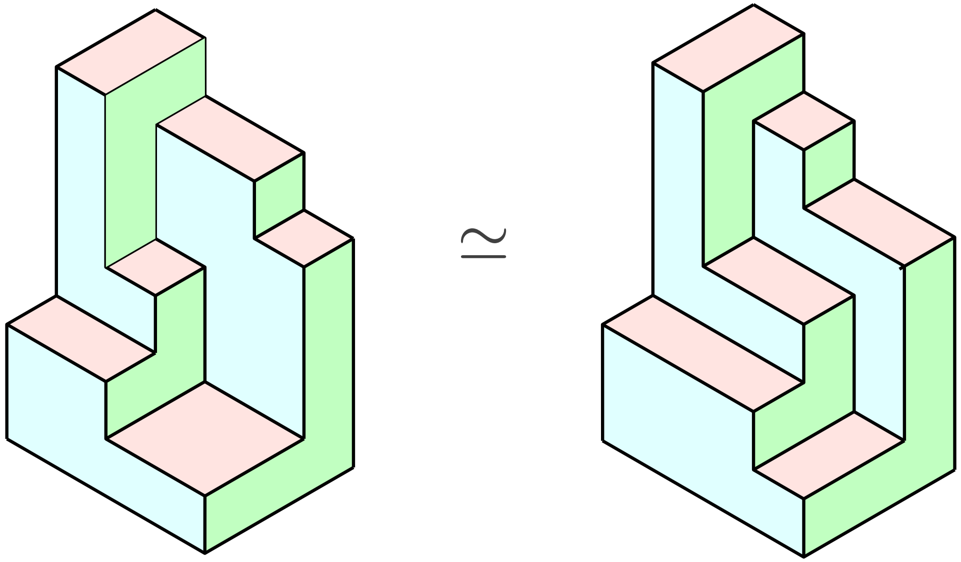
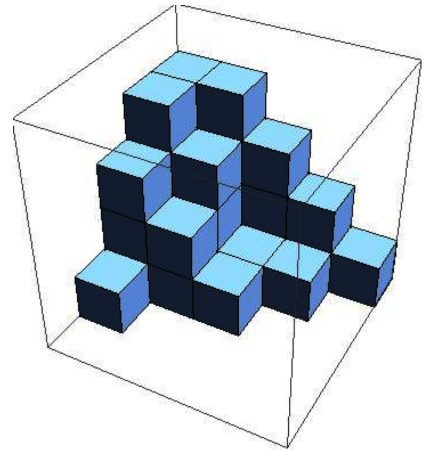


$\neq$ counting plane partitions by volume



[MacMahon'1896]

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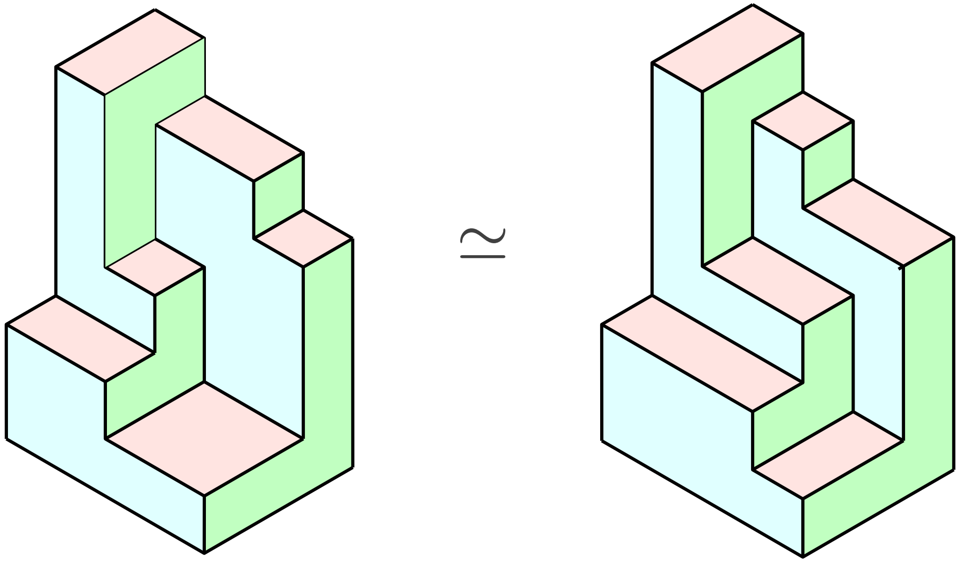

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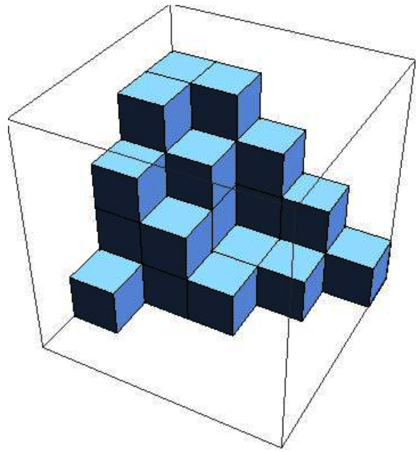

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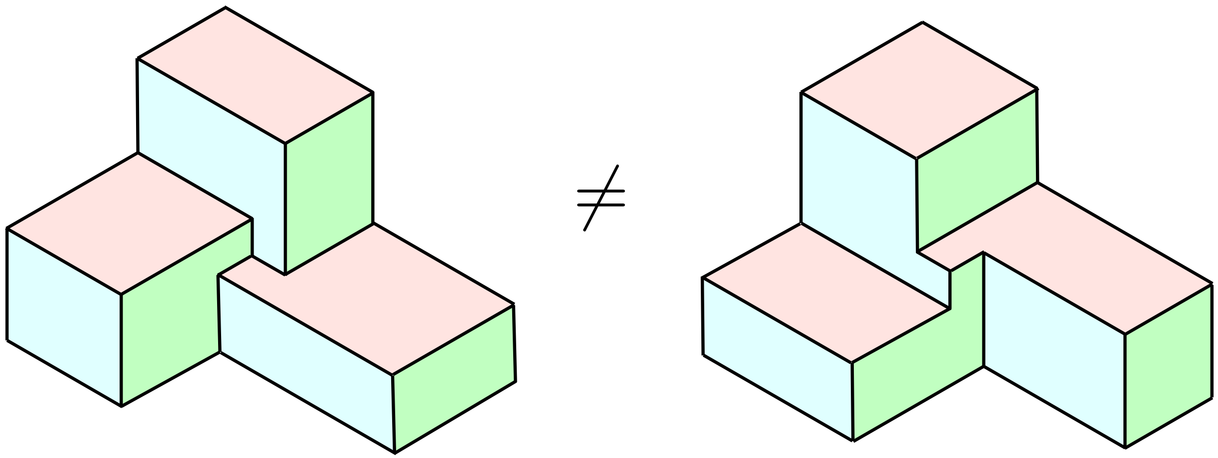

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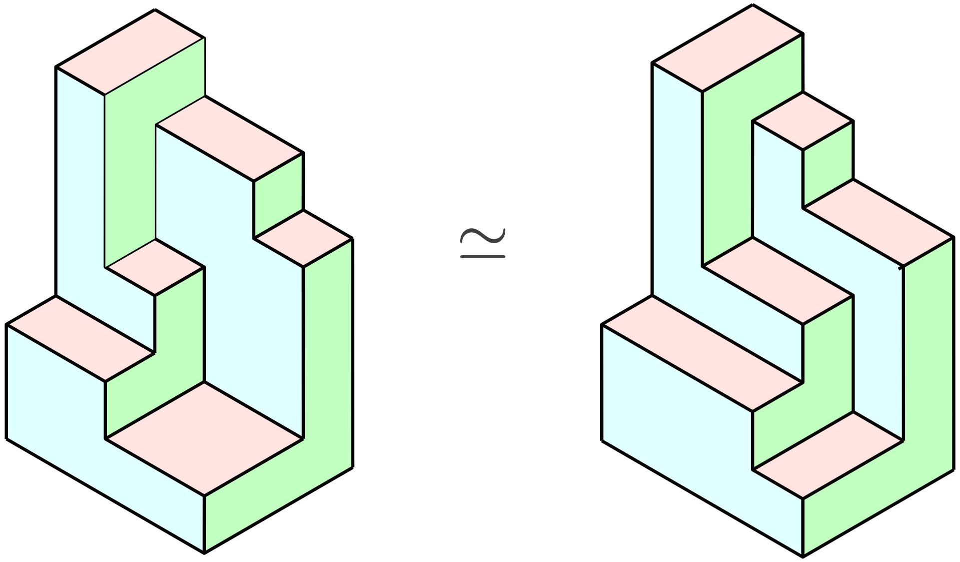

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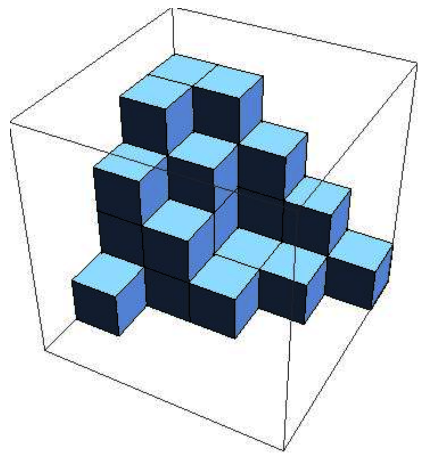

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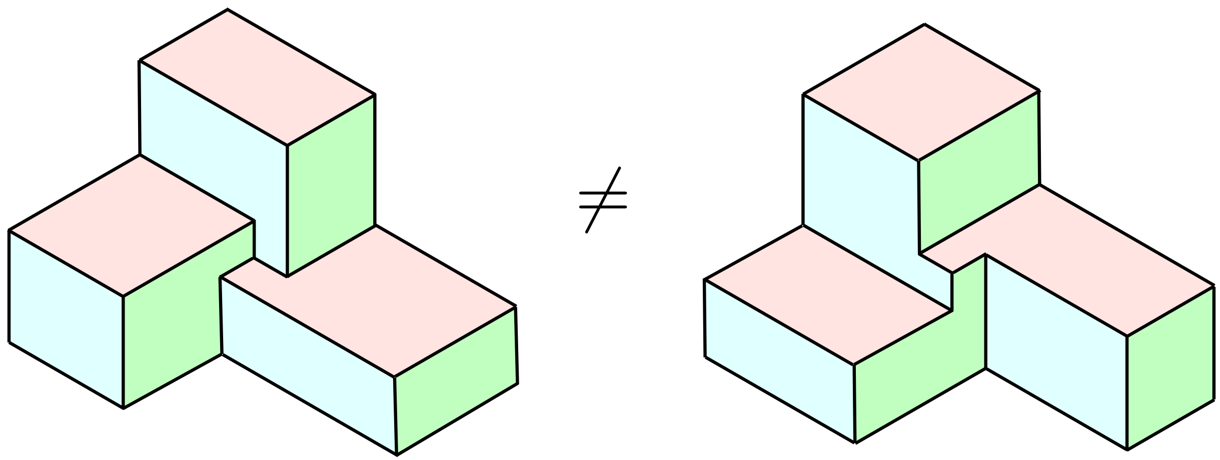
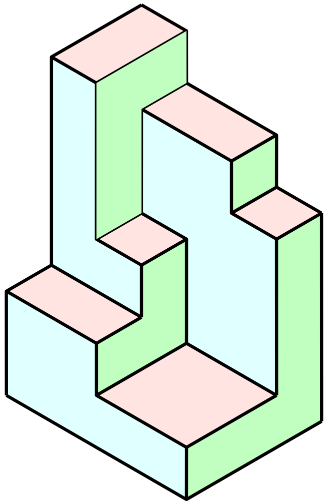
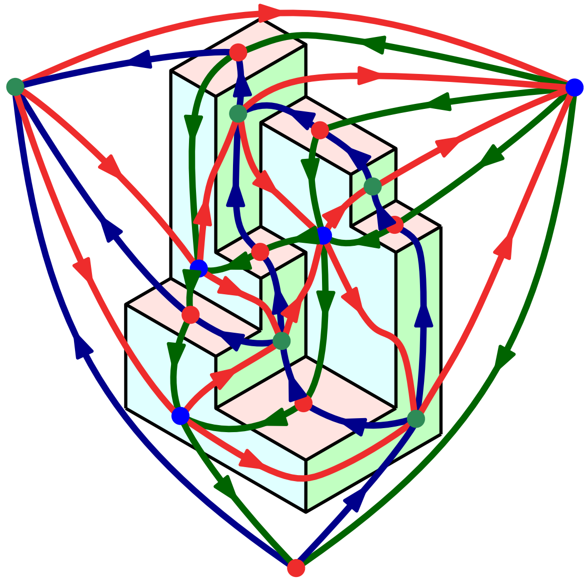


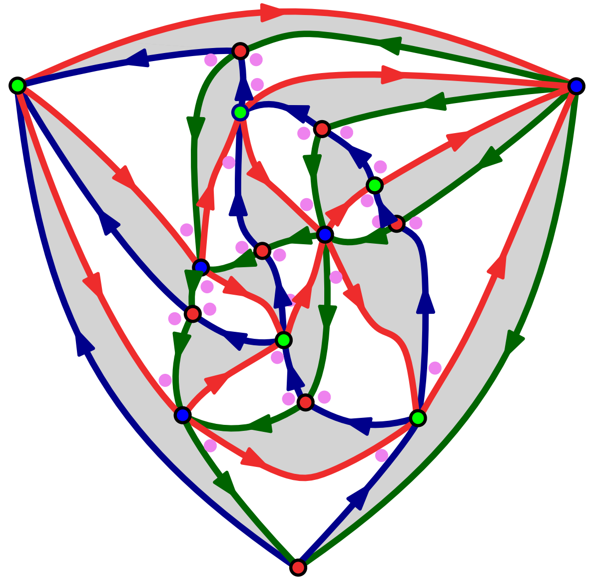
Q:
- exact counting: formula? recurrence?
- asymptotic estimate?

Encoding by orientations

[Eppstein-Mumford'09]







polyhedral orientation

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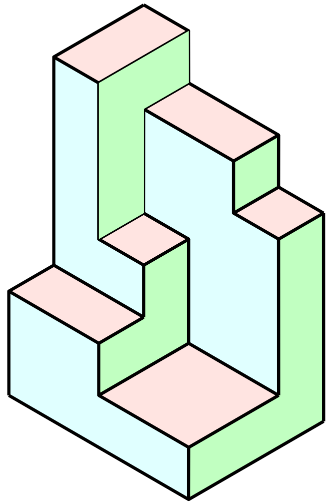
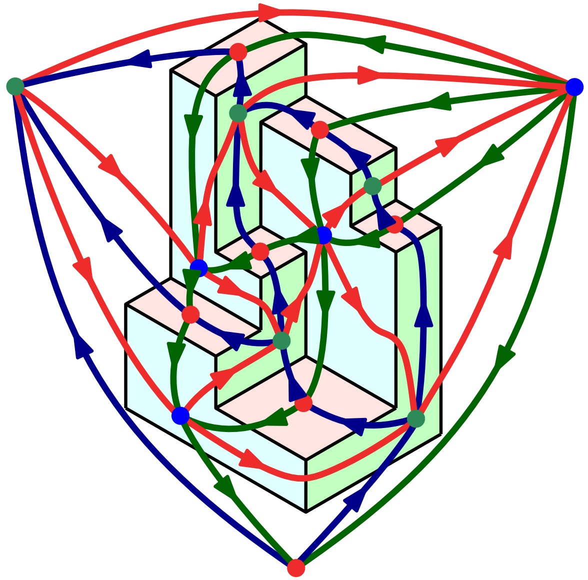
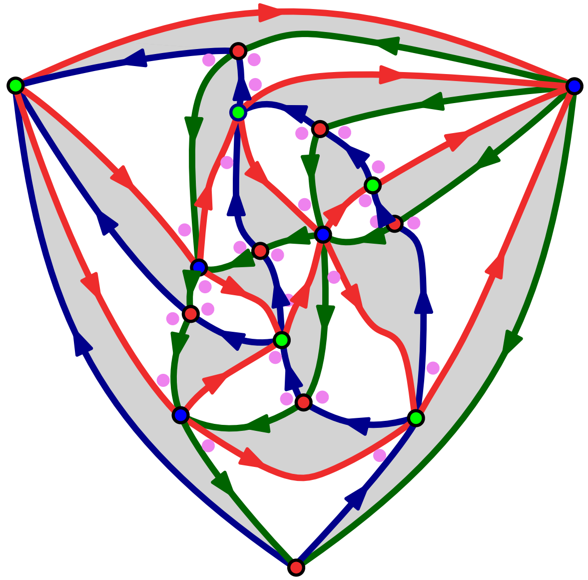
[Eppstein-Mumford'09]

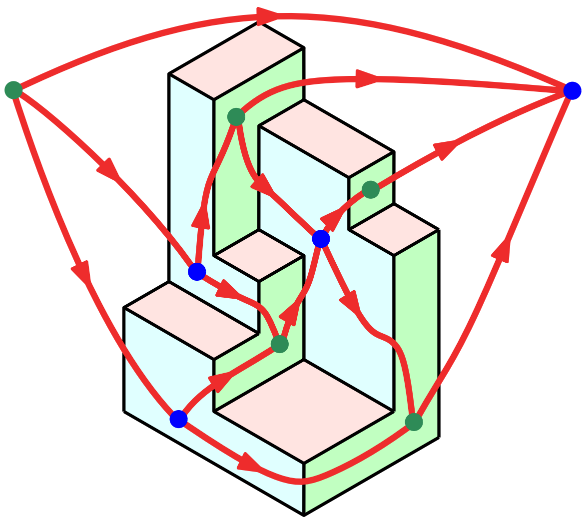


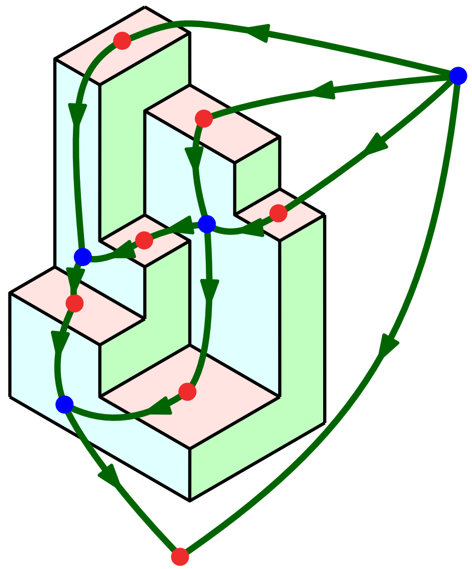


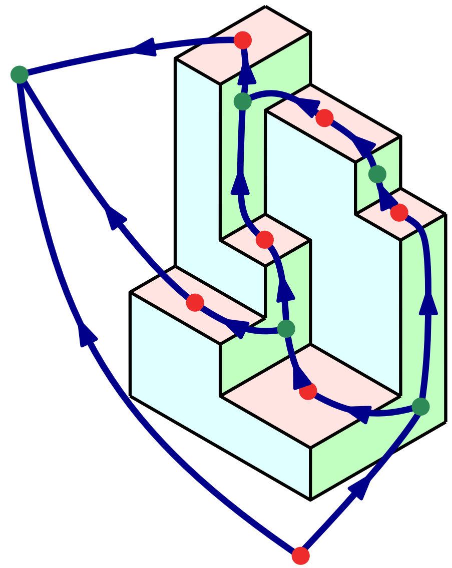


polyhedral orientation

$\Rightarrow$ 3 plane bipolar orientations



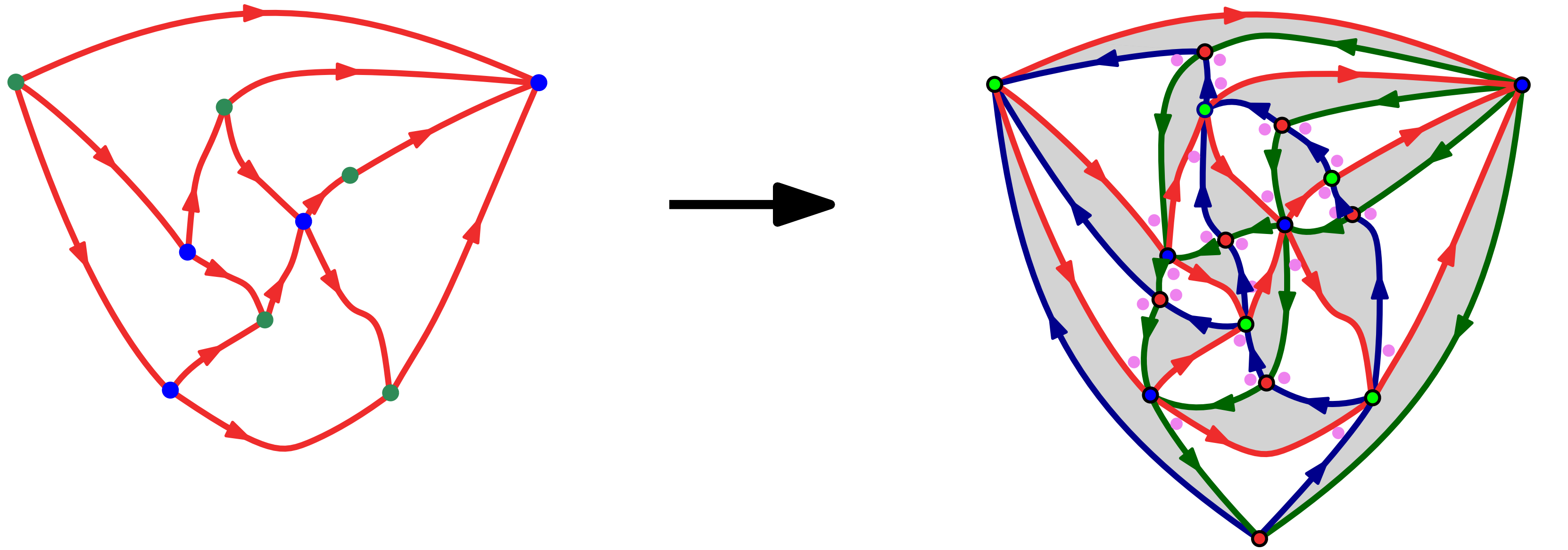




One bipolar orientation is sufficient

[F,Narmanli,Schaeffer'23]

Polyhedral orientation can be reconstructed from red bipolar orientation



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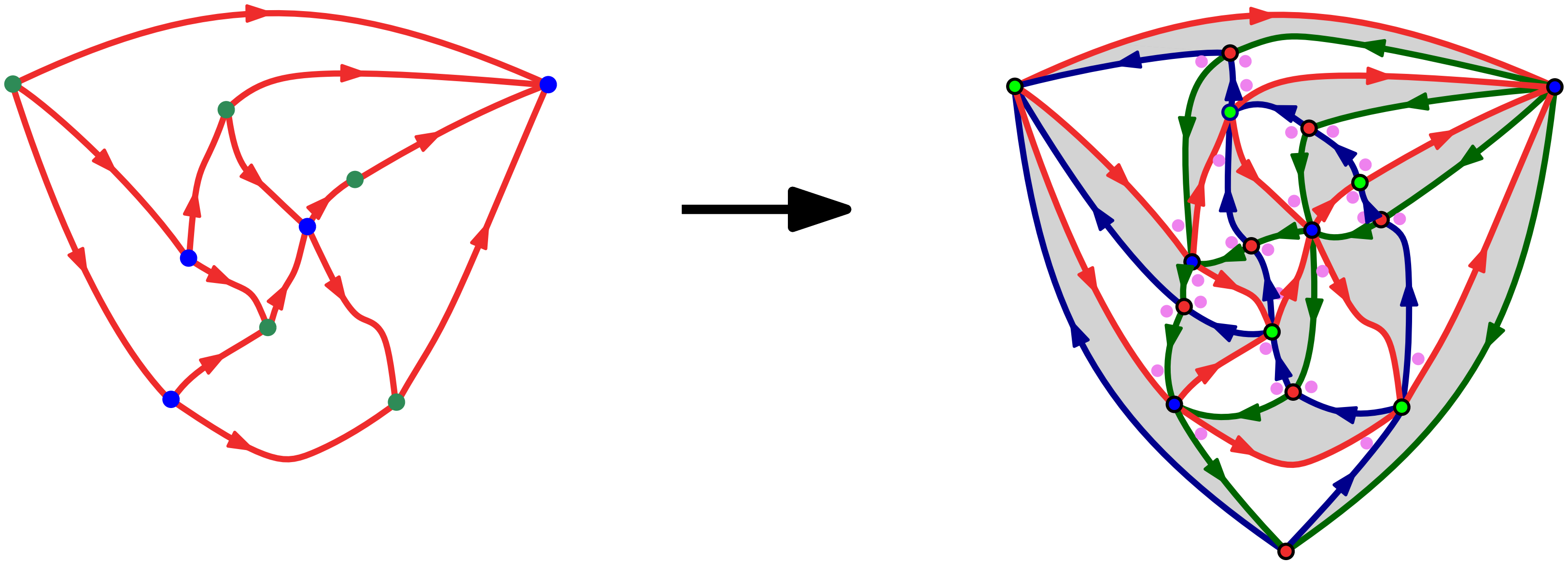

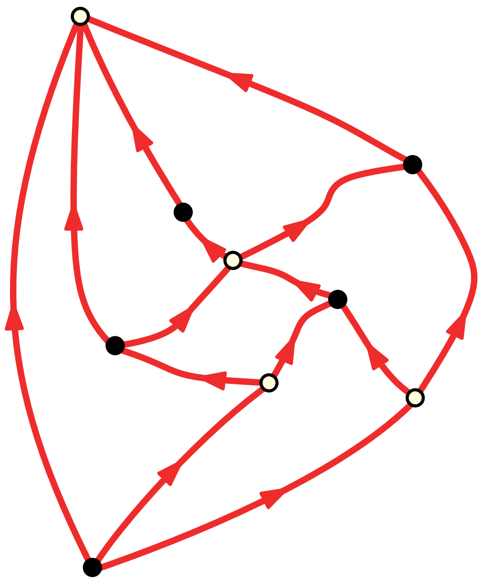

Characterization:

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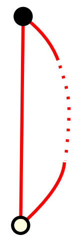
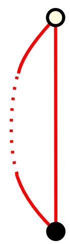
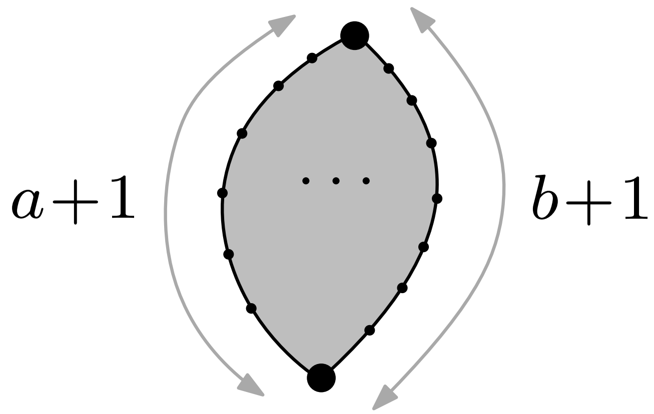
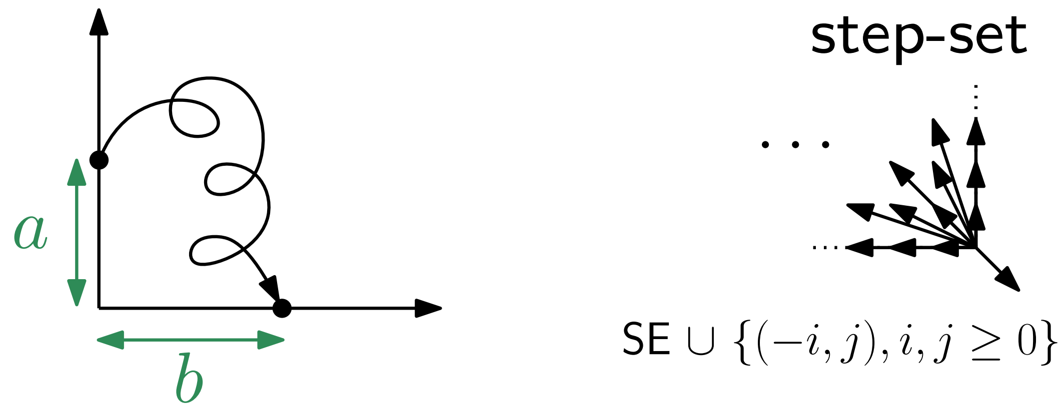
- avoids  and 

Encoding bipolar orientations by quadrant walks

[Kenyon, Miller, Sheffield, Wilson'15]

Plane bipolar orientations ⟷ "Tandem walks" in the quadrant





n edges ⟷ length $n-1$

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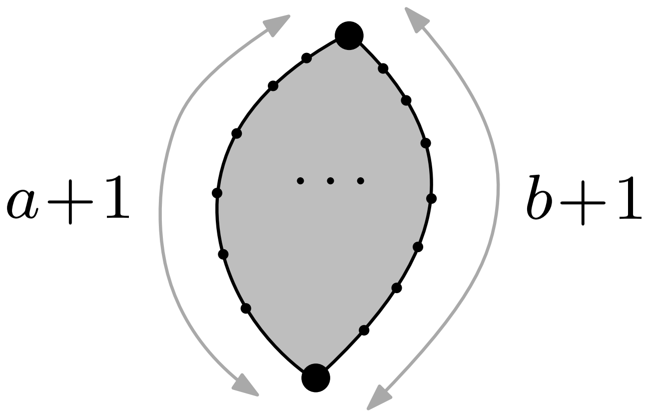
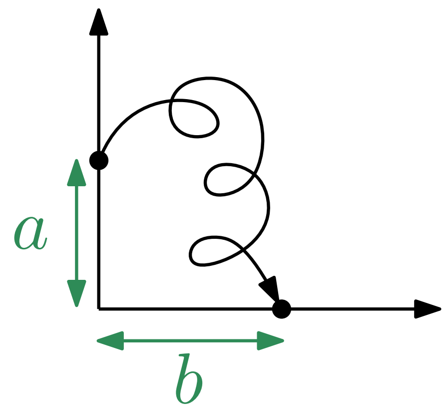
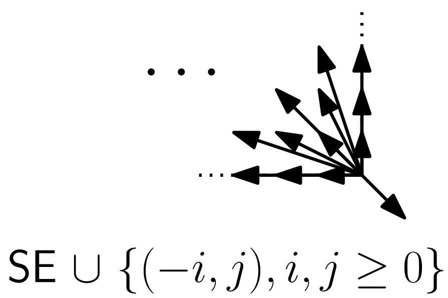
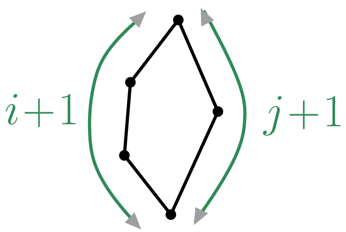






n edges ⟷ length $n-1$

face

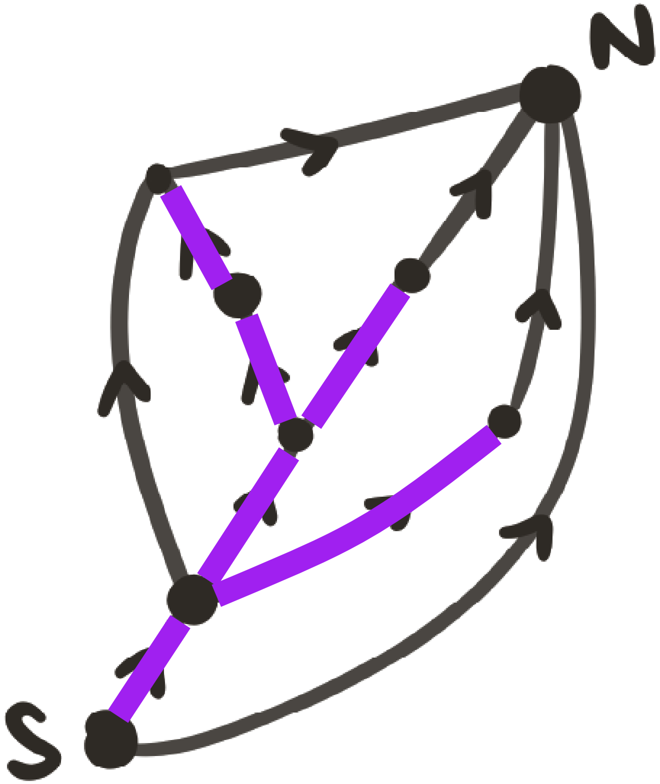


⟷ face-step $(-i,j)$

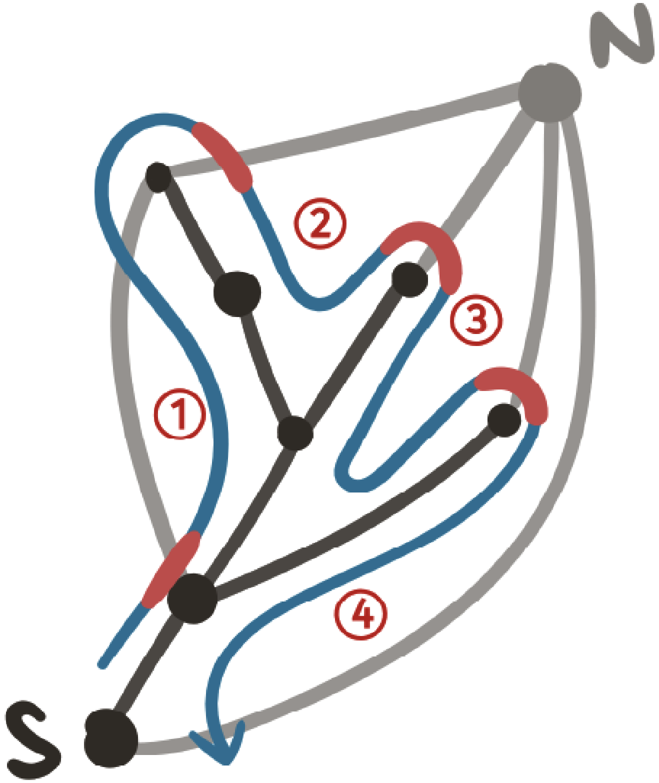
non-pole vertex ⟷ SE step

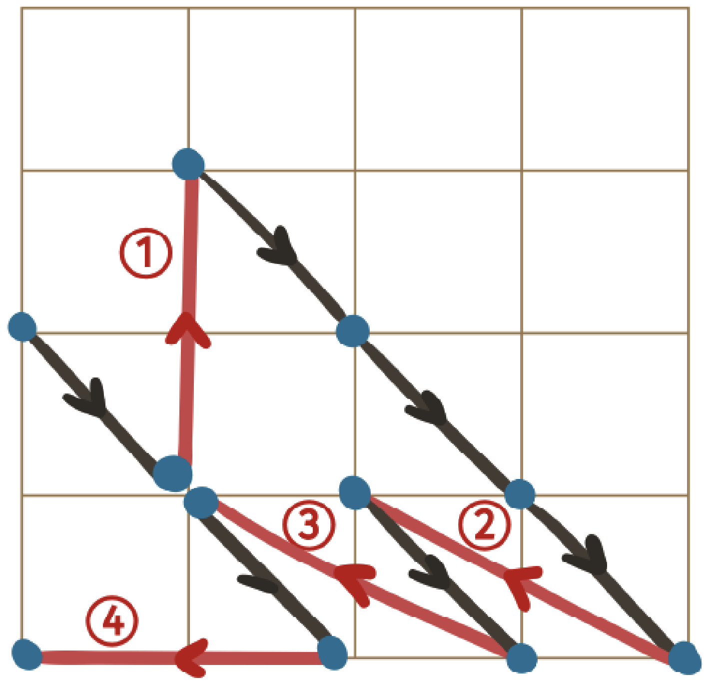
KMSW bijection

From bipolar orientation to tandem walk

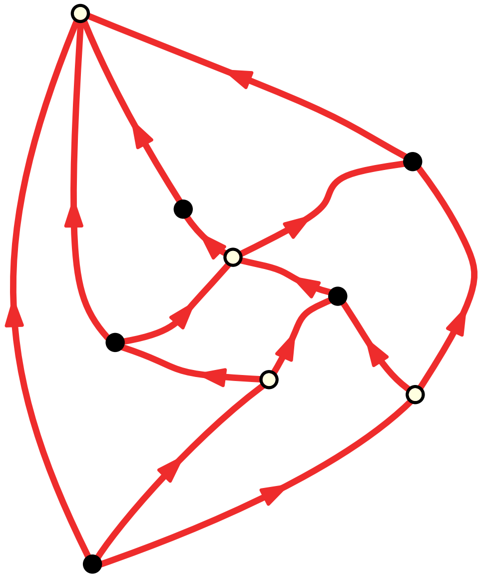


tree of rightmost
incoming edges





Specialization to the red bipolar orientations



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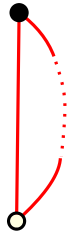
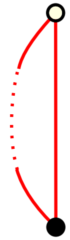
- avoids  and 

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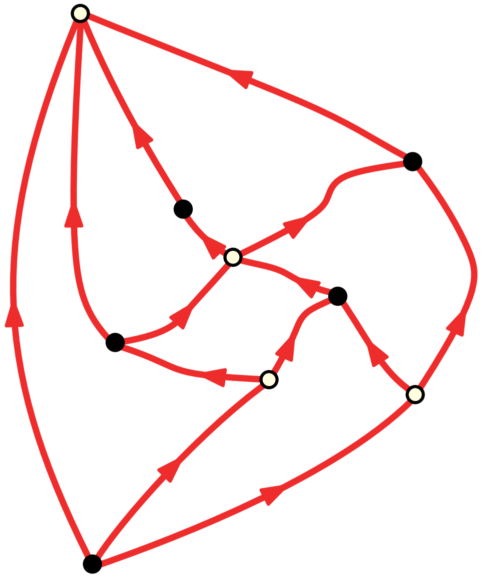

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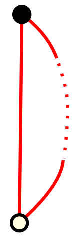
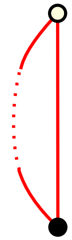
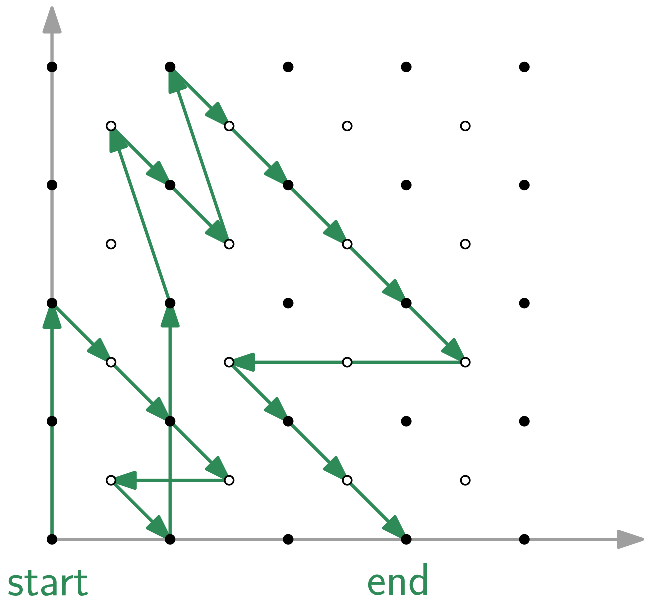
- avoids  and 



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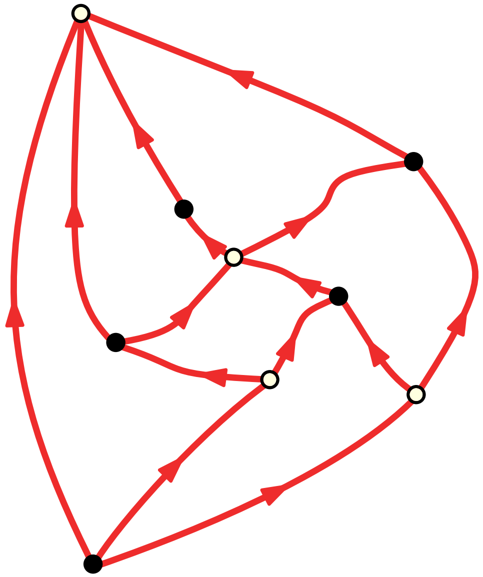

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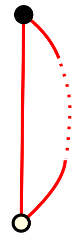
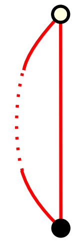
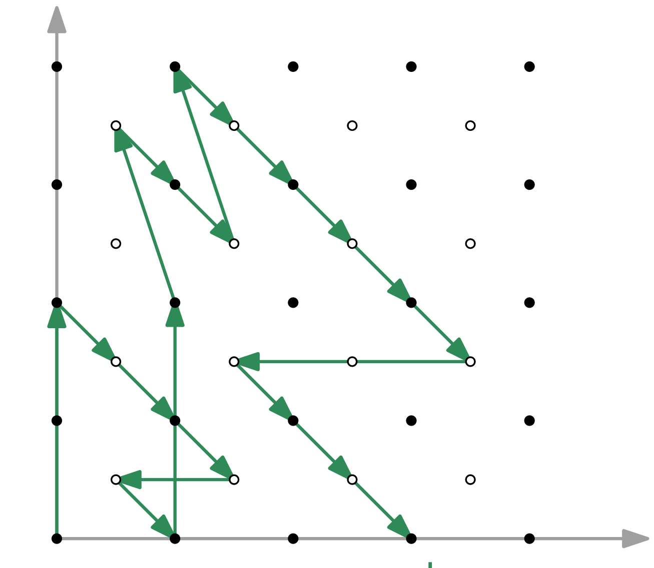
- avoids  and 



- starts at 0, ends on x-axis

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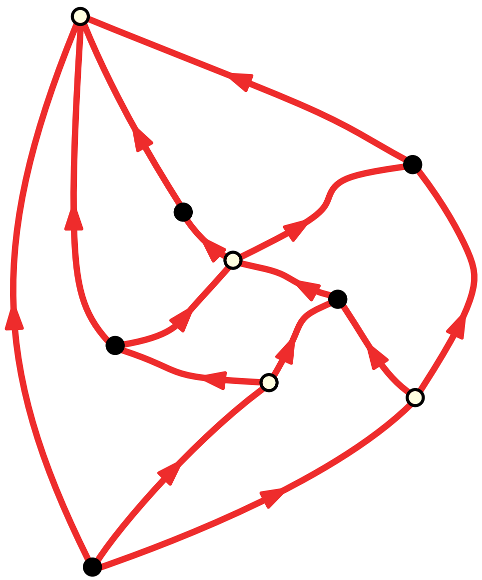

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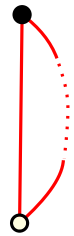
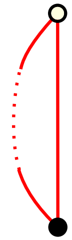
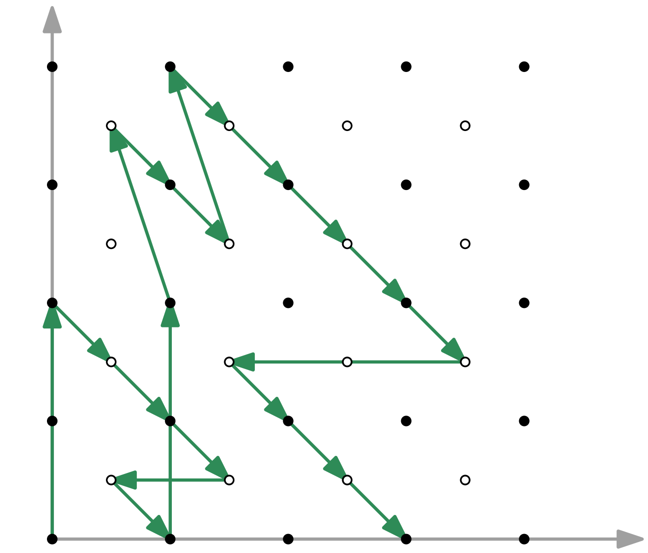
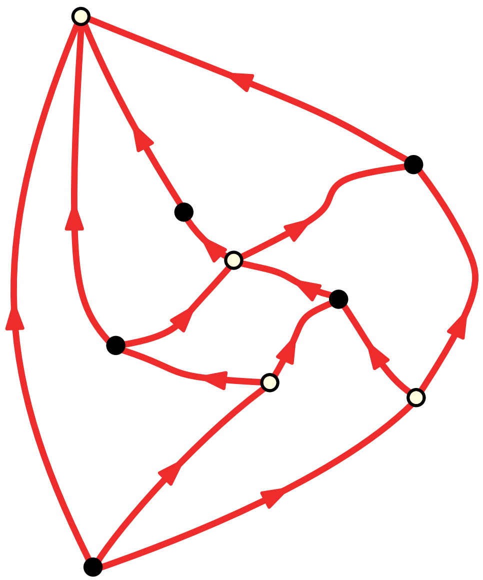
- avoids  and 



- starts at 0, ends on x-axis
- visits only points with $x + y$ even

Specialization to the red bipolar orientations



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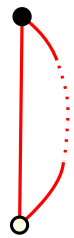
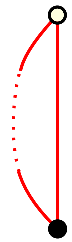
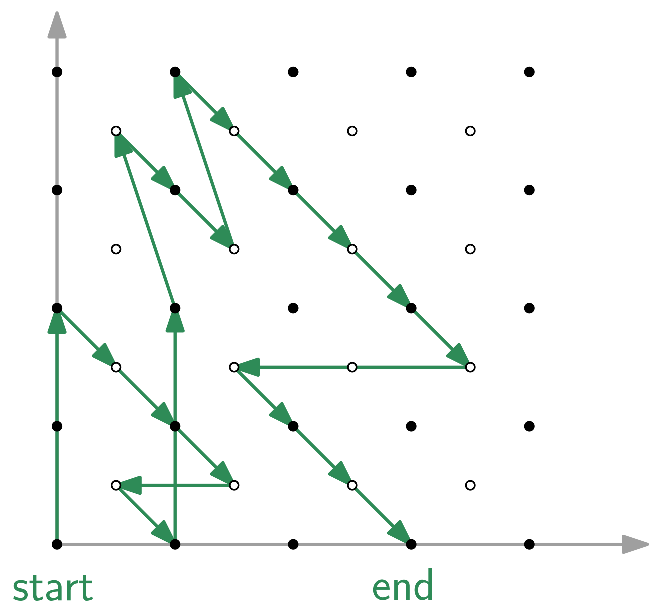
- avoids  and 



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- visits only points with $x + y$ even
- no horizontal step starting from ●
- no vertical step starting from ○

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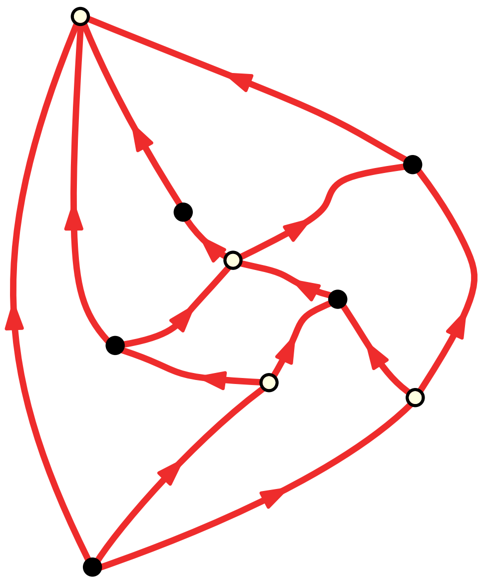

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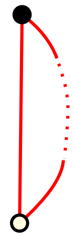
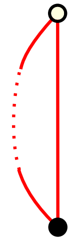
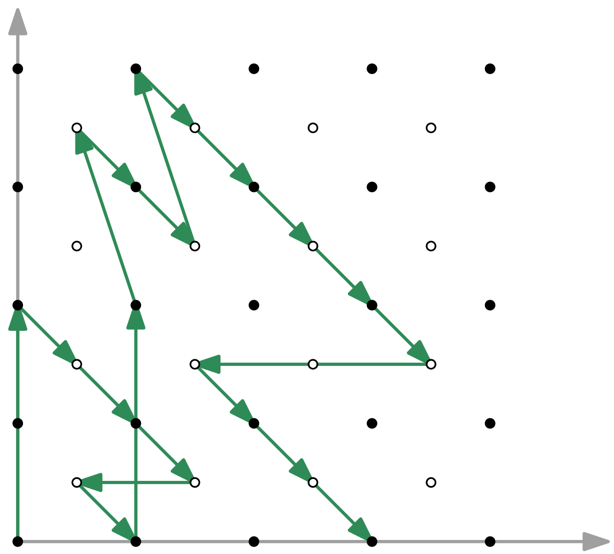
- avoids  and 



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(bimodal effect)

Exact counting: recurrence

By **last step removal**, obtain **recurrence** to compute p_n

($p_n = \sum_{i \geq 0} a_n(i, 0)$, with recurrence on $a_n(i, j)$)

$$\sum_{n \geq 1} p_n t^n = t^3 + 3\,t^5 + 4\,t^6 + 15\,t^7 + 39\,t^8 + \mathbf{122}\,t^9 + 375\,t^{10} + 1212\,t^{11} + \cdots$$

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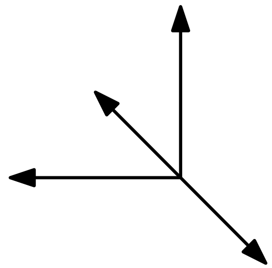
Similarly, can obtain recurrence for $p_{a,b,c} = \#$ corner polyhedra with
a blue flats, b red flats, c green flats

$$\sum_{a,b,c\geq 1} p_{a,b,c} u^a v^b w^c = uvw + (u^2v^2w + uv^2w^2 + u^2vw^2) + 4u^2v^2w^2$$
$$+(u^3v^3w + 4u^3v^2w^2 + 4u^2v^3w^2 + u^3vw^3 + 4u^2v^2w^3 + uv^3w^3) + \cdots$$

Asymptotic bounds for excursions in quadrant

General method (saddle bound), e.g. for $\mathcal{S} =$ 

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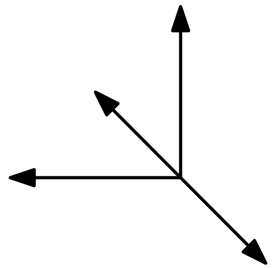
General method (saddle bound), e.g. for $\mathcal{S} =$ 

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Then $\forall x, y > 0$, $\quad \sum_{i,j \in \mathcal{Z}^2} a_n(i,j) x^i y^j = S(x,y)^n$

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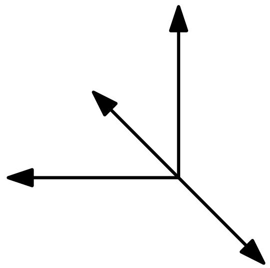
General method (saddle bound), e.g. for $\mathcal{S} =$ 

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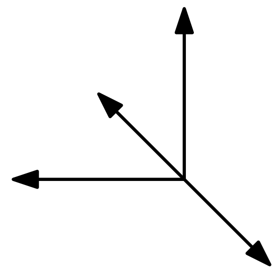

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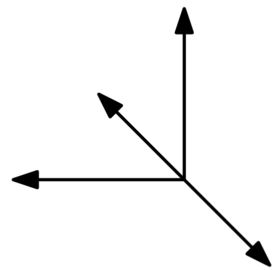

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Rk: optimal $(x,y) \quad \leftrightarrow \quad (x,y)$-weighted random $\mathcal{S}$-walk has drift$= 0$

each step $s=(i,j)\in\mathcal{S}$ has proba $\quad \frac{x^iy^j}{S(x,y)}$

Asymptotic results for corner polyhedra

- **Growth rate**: $\lim_n (p_n)^{1/n} = 9/2$
- **Conjecture**: $p_n \sim c\,(9/2)^n\, n^{-\alpha}$ where $c > 0$

$$\alpha = 1 + \frac{\pi}{\arccos(9/16)} \approx 4.23$$

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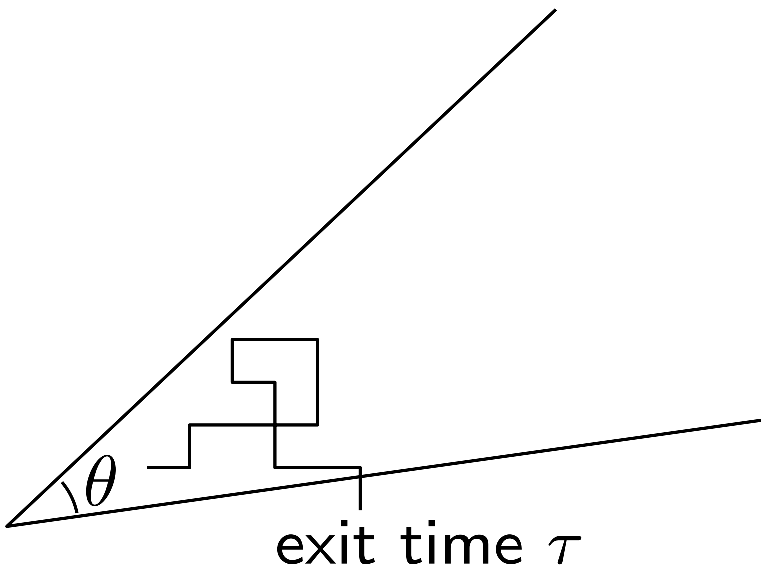
Explanation:

reduction to Denisov and Wachtel'15 "random walks in cones"



$$\mathbb{P}(\tau > n) \sim c'\; n^{-\frac{\pi}{2\theta}}$$

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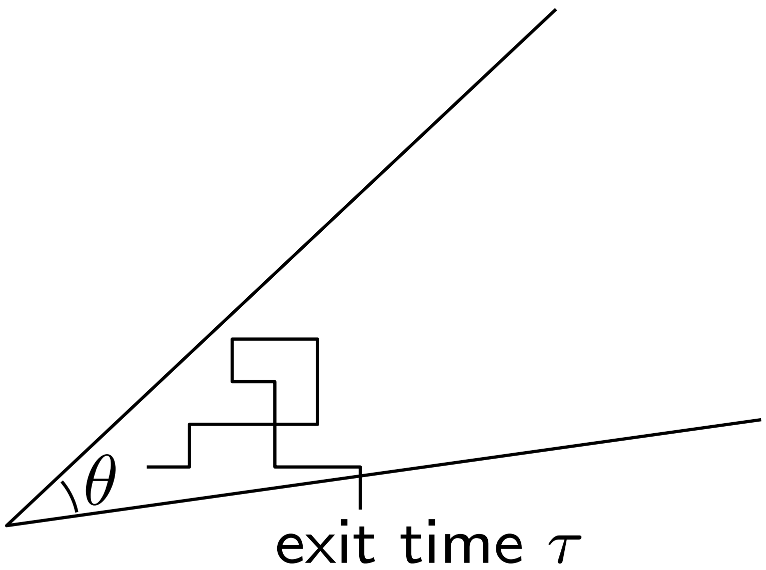

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$$\alpha = 1 + \frac{\pi}{\arccos(9/16)} \approx 4.23$$

Rk: Conjecture would imply $\sum_n p_n z^n$ **not D-finite** (since $\alpha \notin \mathbb{Q}$)

criterion in **[Bostan, Raschel, Salvy'14]**

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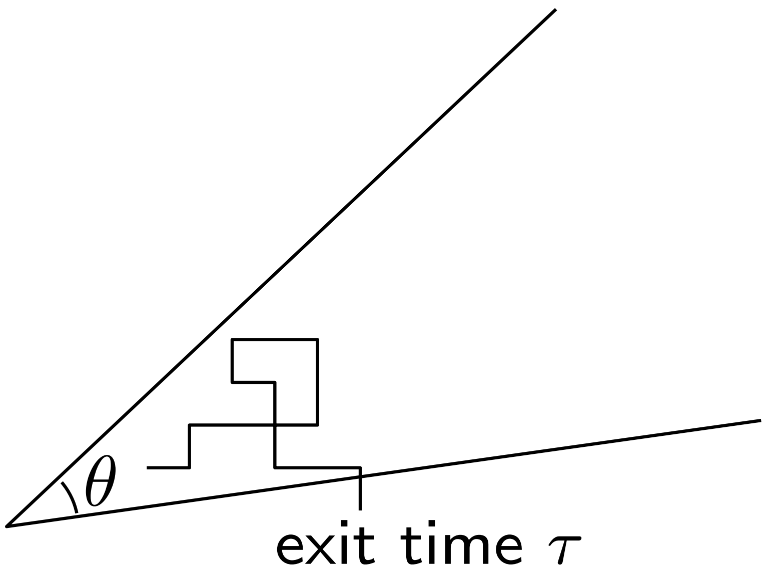
reduction to Denisov and Wachtel'15 "random walks in cones"



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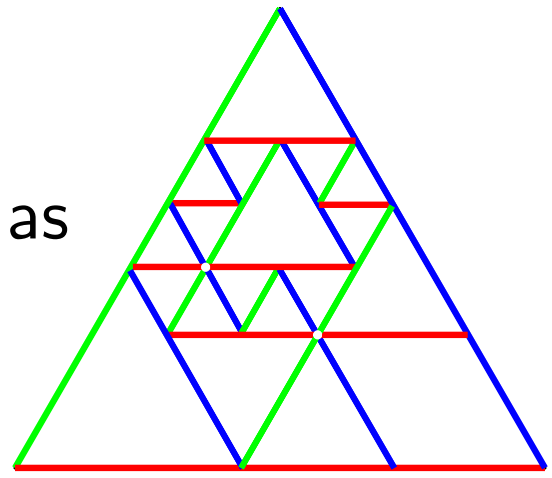
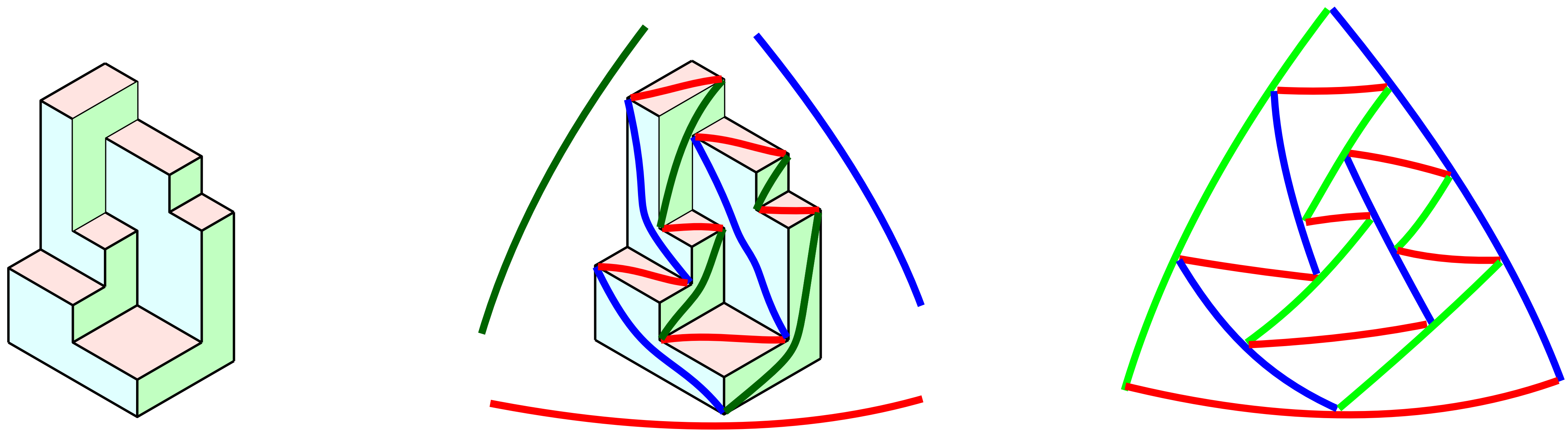
Relation to some tricolored contact-systems

[Gonçalves'19]

every **corner triangulation** has a unique **tricolored segment-contact** representation as

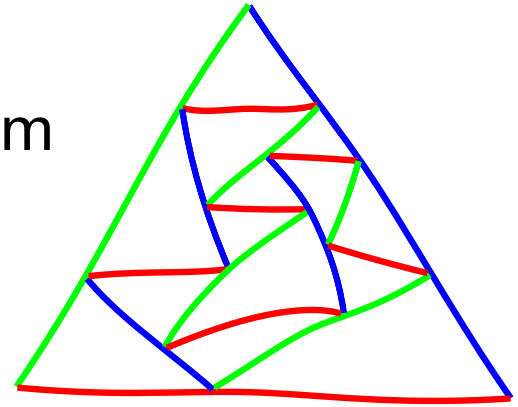


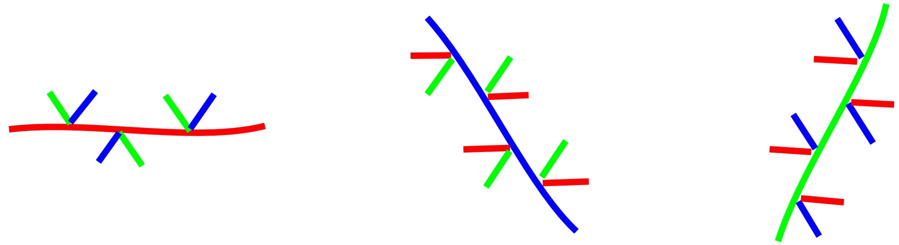
Corner polyhedra (types) can be encoded bijectively by such a **topological tricolored contact-system** of (smooth) curves

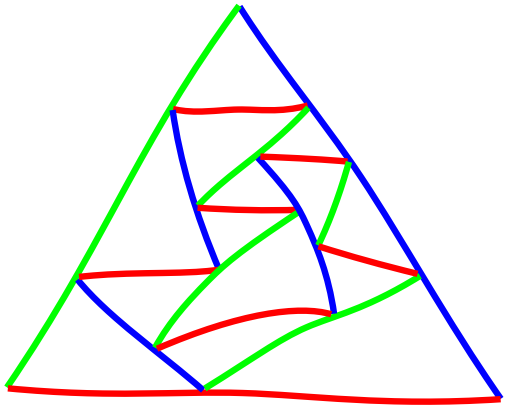


2 ways of counting tricolored contact-systems

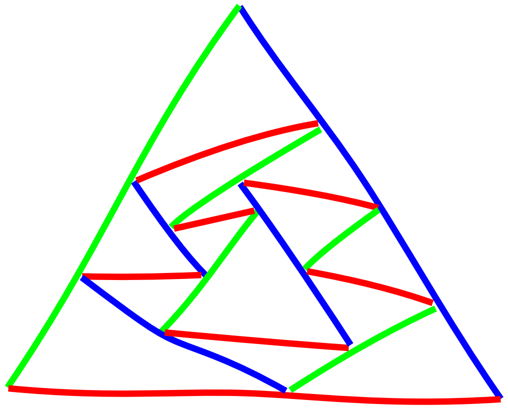
Contact-system of curves



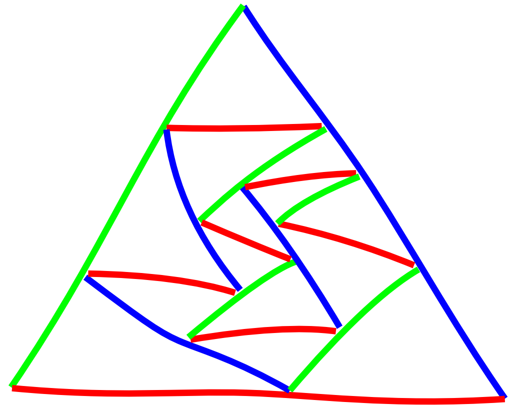




$\simeq$ strong



$\simeq$ weak

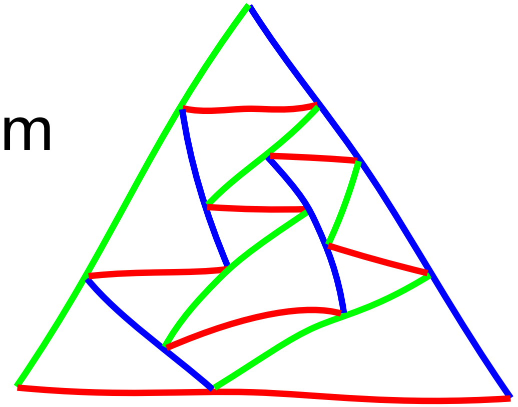


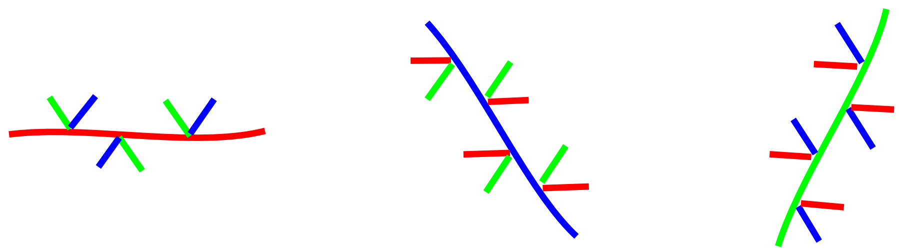
$w_n =$ # weak equivalence classes with $2n$ regions

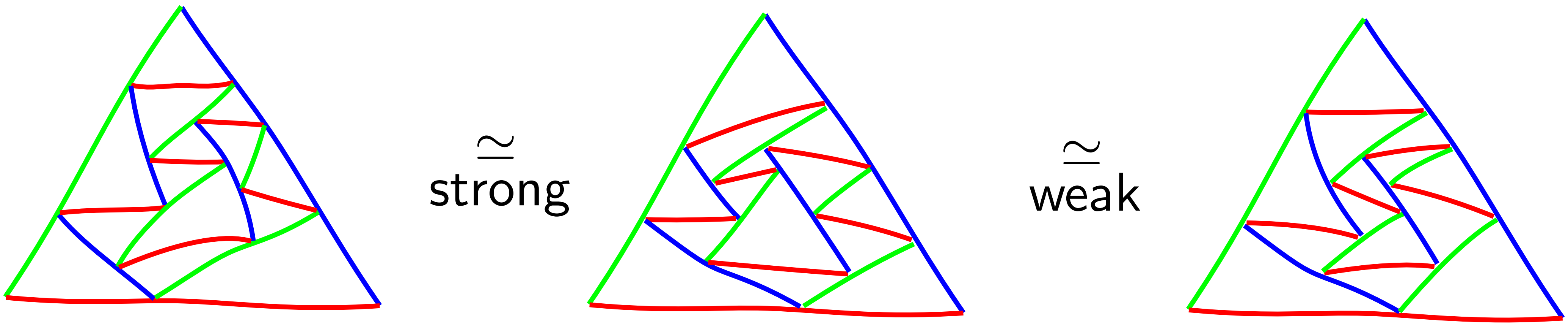
$s_n =$ # strong equivalence classes with $2n$ regions

2 ways of counting tricolored contact-systems

Contact-system of curves







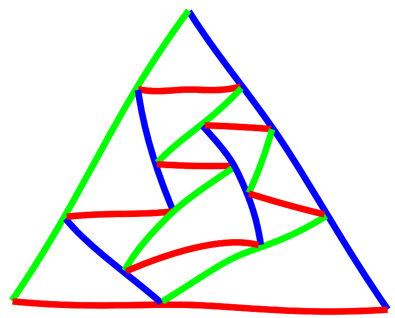
$w_n =$ # weak equivalence classes with $2n$ regions $= p_n$

$s_n =$ # strong equivalence classes with $2n$ regions

Asymptotic enumeration

Asymptotic estimate $c\,\gamma^n\,n^{-\alpha}$

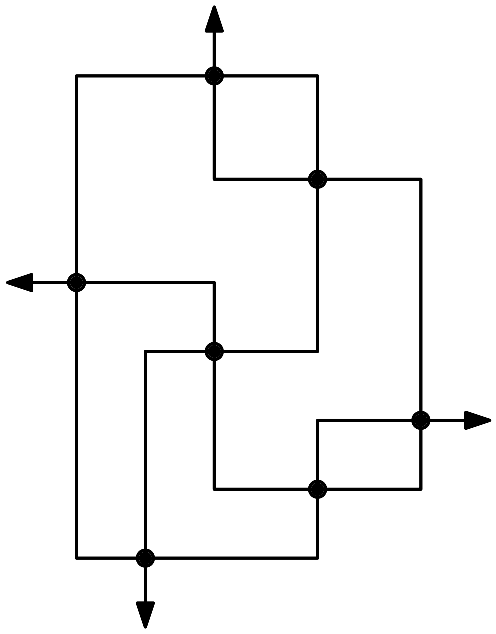
$\alpha = 1 + \frac{\pi}{\theta}$



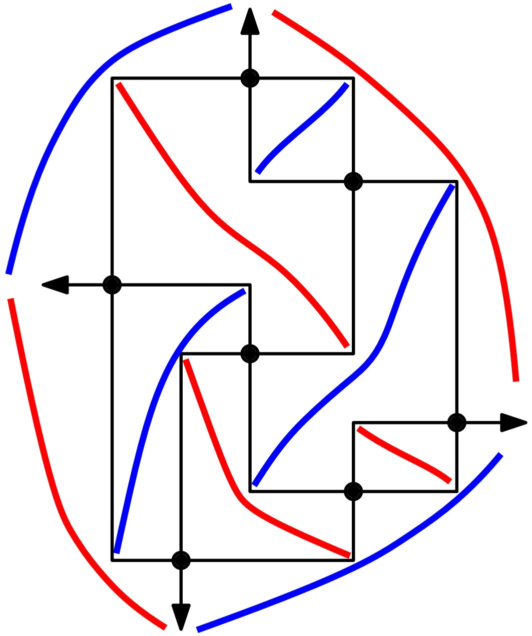
	weak	strong
	polyhedral orientations	(3c) Schnyder labelings
γ	$9/2$	$16/3$
$\cos(\theta)$	$9/16$ (*)	$22/27$ (*)
α	$\approx 4.23 \notin \mathbb{Q}$	$\approx 6.08 \notin \mathbb{Q}$

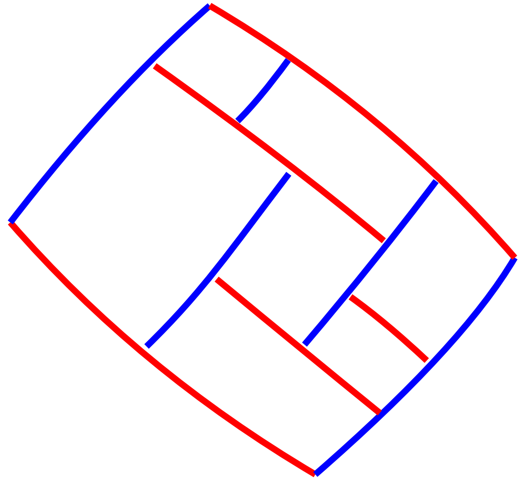
(*) up to extending [Denisov-Wachtel] to bimodal setting

Similar models in 2d with 2 colors

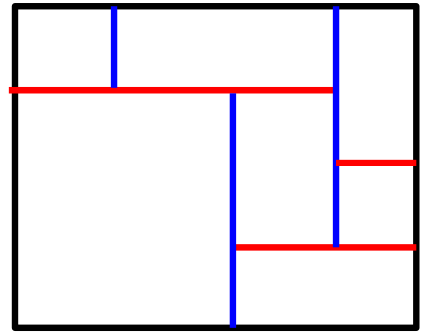


1-bent orthogonal drawing



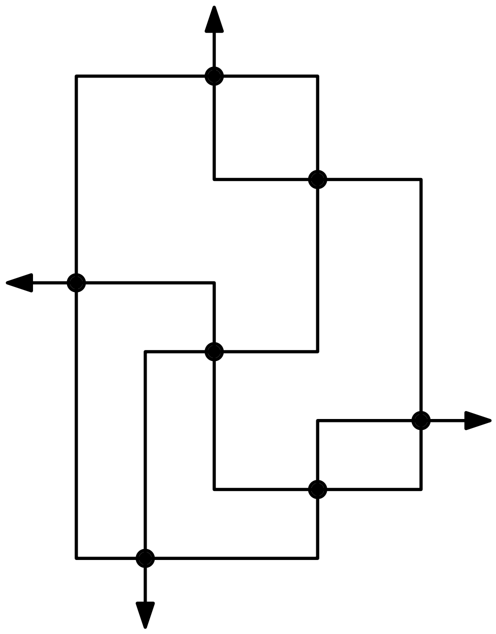


2-colored contact-system

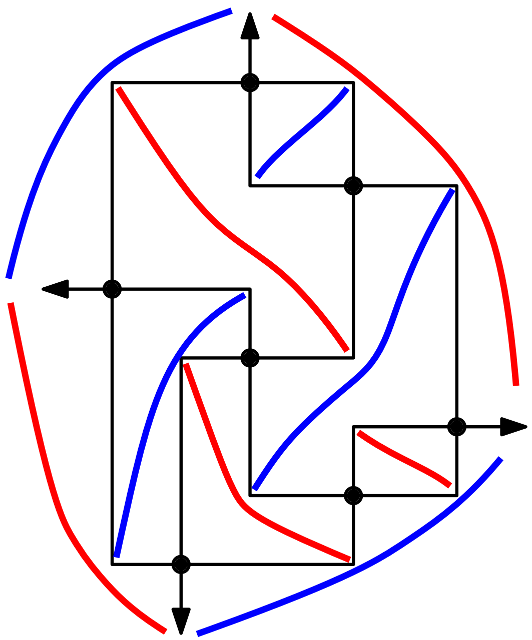


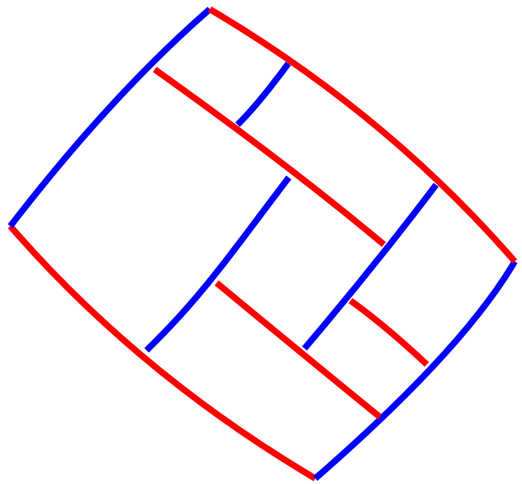
rectangulation

Similar models in 2d with 2 colors

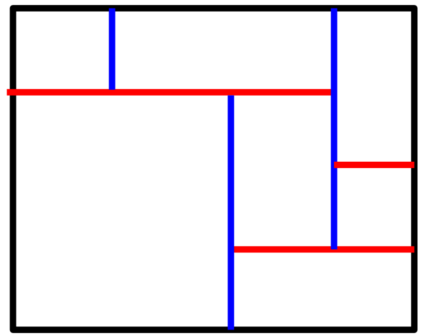


1-bent orthogonal drawing

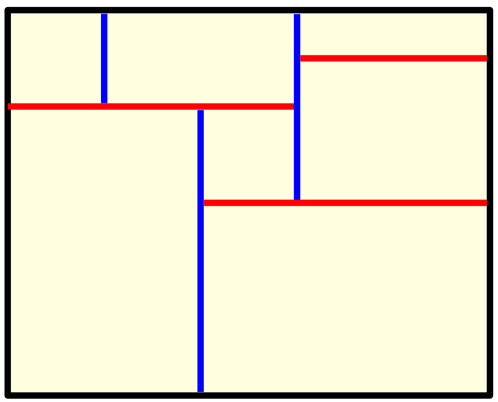




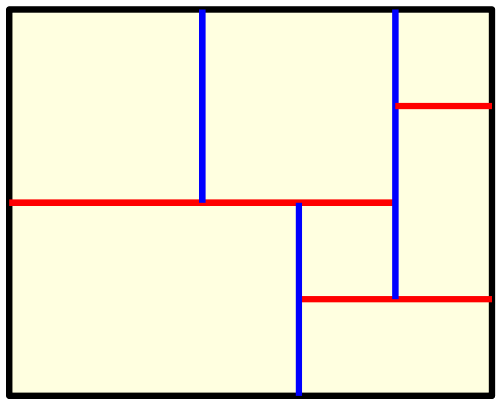
2-colored contact-system



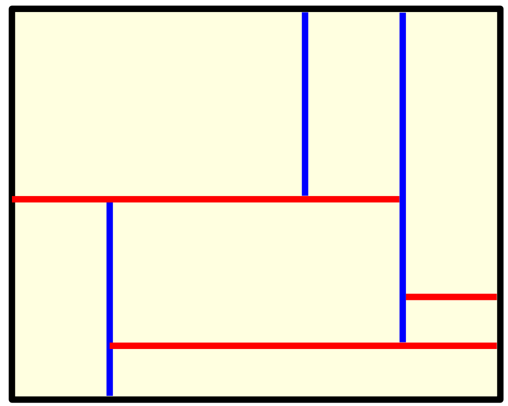
rectangulation



$\underset{\text{strong}}{\simeq}$



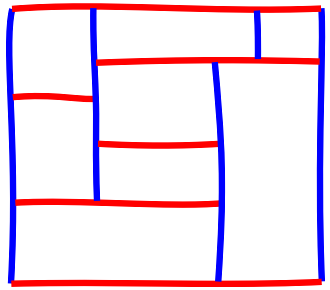
$\underset{\text{weak}}{\simeq}$

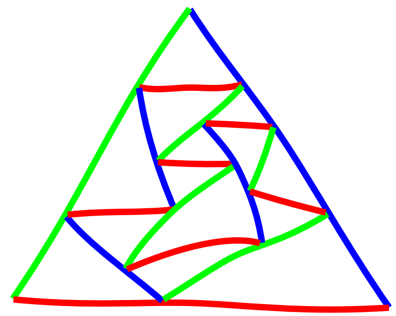


Summary on asymptotic enumeration

Asymptotic estimate $c\, \gamma^n\, n^{-\alpha}$

$\alpha = 1 + \frac{\pi}{\theta}$

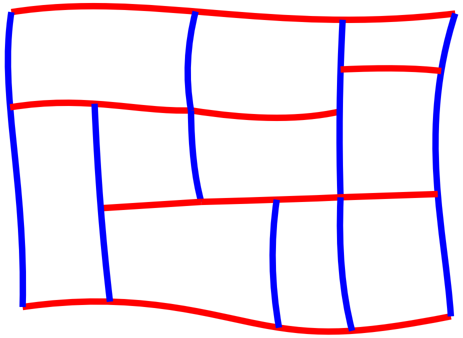


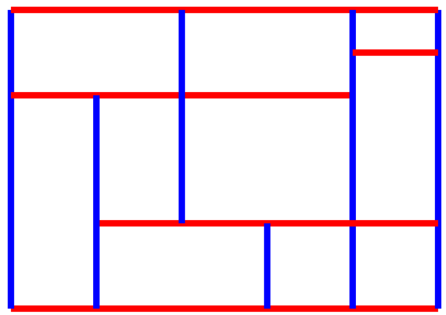


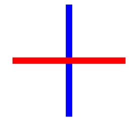
	weak	strong	weak	strong
	bipolar orientations	transversal structures	polyhedral orientations	(3c) Schnyder labelings
γ	8	$27/2$	$9/2$	$16/3$
$\cos(\theta)$	$1/2$	$7/8$	$9/16$ (*)	$22/27$ (*)
α	4 Baxter (D-finite)	$\approx 7.21 \notin \mathbb{Q}$	$\approx 4.23 \notin \mathbb{Q}$	$\approx 6.08 \notin \mathbb{Q}$

(*) up to extending [Denisov-Wachtel] to bimodal setting

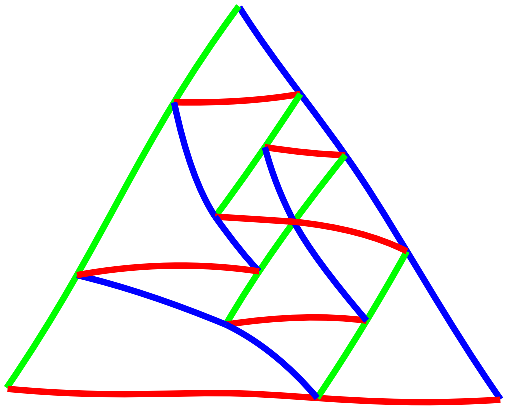
Extension to models with degeneracies

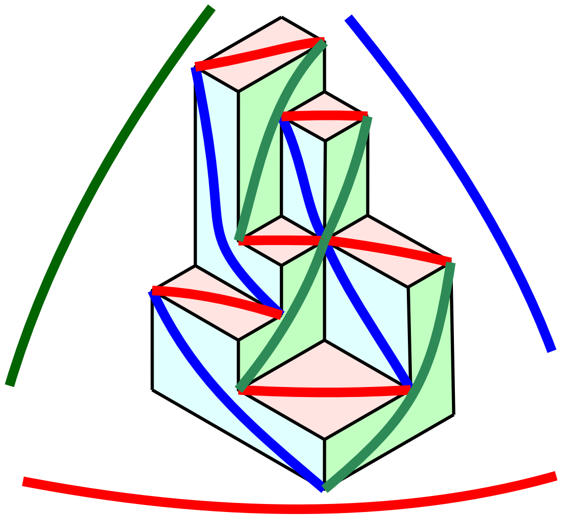


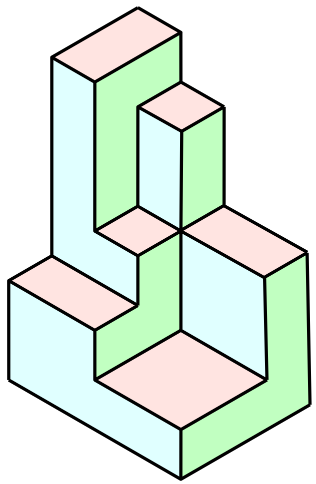


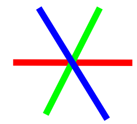
weight v per 

also counted in [Conant,Michaels'12]

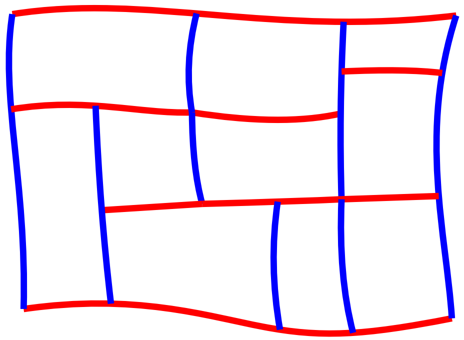


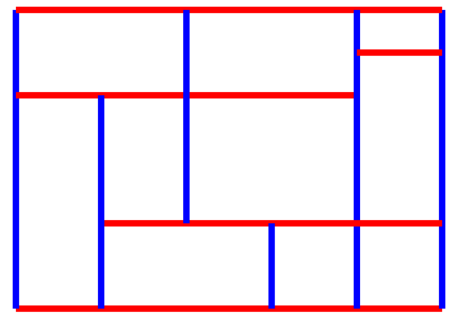


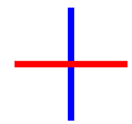


weight v per 

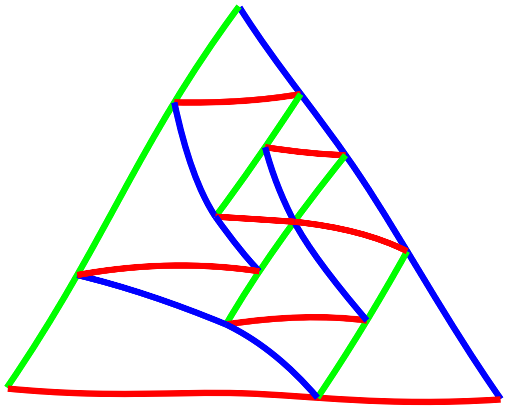
Extension to models with degeneracies

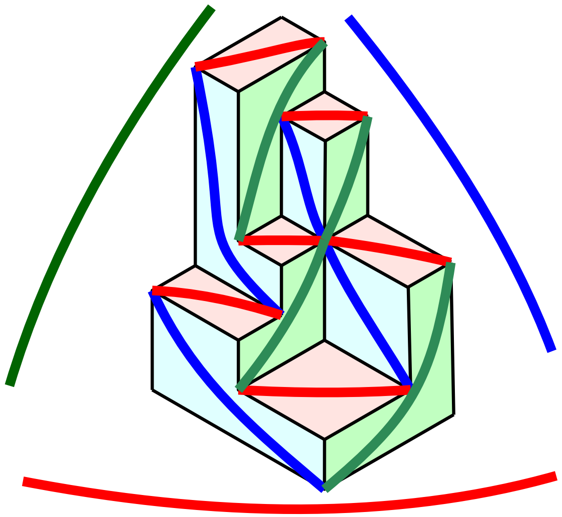


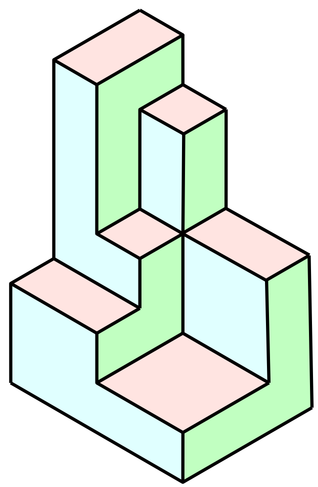


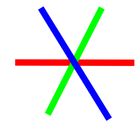
weight v per 

also counted in [Conant,Michaels'12]







weight v per 

Asymptotic exponent $\alpha(v)$ computable $\alpha(v) \to \infty$ as $v \to \infty$

regular grid behaviour