Game Complexity: Between Geography and Santorini

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Joint work with Carson Geissler
Santorini

Game End
- Lose if can’t move
- Win if move onto height 3

Move
- Adjacent square
  - Diagonals count
    - Can’t move more than 1 square

Build
- Increase height of adjacent square (after move)

Worker
Tower win

Partizan Combinatorial Game

- 2 players
- No hidden information or randomness
- Game must end

Adjustable Parameters

Underlying Board
- Grid or general graph?
- Directed or undirected?

Game Rules
- “Tower win” rule?
- Number of workers?
- Only build where you came from?
- Restriction on moving upward?
- Shared workers?
“Easy”: Can describe a winning strategy concisely.
(Formally: In $\text{P}$.)

“Hard”: Known or not believed to have a consice winning strategy.
(Formally: $\text{PSPACE}$-hard.)
Comparing Complexities: Reductions

Game $A$ is at least as hard as Game $B$ if you can simulate $B$ using the rules of $A$. 
Starting Point: TQBF Game

\[ \phi = (x_1 \lor \neg x_2 \lor x_3) \land (x_1 \lor x_3 \lor \neg x_4) \land \ldots \]

Setup: 3-CNF Boolean formula \( \phi \) with variables \( x_1, x_2, \ldots, x_{2n} \)

Rules: Players take turns setting variables in order \( x_1, x_2, \ldots, x_{2n} \).

End of Game: First player wins iff \( \phi \) is true.

\[ \exists x_1 \forall x_2 \exists x_3 \ldots \forall x_{2n} \phi \]

Classical “hard” game

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Basic Santorini Reductions

- General graphs are at least as hard as grids.
- Directed graphs are at least as hard as undirected graphs.
- Games with additional workers are at least as hard as games without.
- Higher max heights are at least as hard as lower max heights.
Basic Santorini Reductions

- General graphs are at least as hard as grids.
- Directed graphs are at least as hard as undirected graphs.
- Games with additional workers are at least as hard as games without.
- Higher max heights are at least as hard as lower max heights.
- Max height 2 is at least as hard as max height 1 without “tower wins”
Our Assumptions Going Forward

- General graphs (directed or undirected)
- No “tower wins”
- 1 worker (shared), or 1 worker per player (impartial vs. partizan)
- Max height 1 unless otherwise stated
- Same upward movement restrictions as Santorini
- “Restricted builds” means build only where moved from
Impartial, Directed, Restricted Builds

Classically "hard" (Schaefer 1978; Lichtenstein, Sipser 1980)

\[ \Phi = (x_1 \lor x_2 \lor x_3) \land (x_1 \lor x_2 \lor x_n) \land \cdots \]

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Impartial, Directed, Restricted Builds

- a.k.a. GEOGRAPHY
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$$\Phi = (x_1 \lor \overline{x_2} \lor x_3) \land (\overline{x_1} \lor x_2 \lor x_{2n}) \land \cdots$$
Impartial, \textit{Undirected}, Restricted Builds

- a.k.a. Undirected GEOGRAPHY
Impartial, *Undirected*, Restricted Builds

- a.k.a. Undirected GEOGRAPHY
- “Easy!” (Fraenkel, Scheinerman, Ullman 1993)
- Strategy: If vertex with worker is involved in every maximum-size matching, choose one and win by moving along matching edges.
Impartial, Undirected, Unrestricted Builds

- “Easy”
- Strategy: If vertex with worker is involved in every maximum-size matching, choose one and win by moving along matching edges and building where moved from. (Same strategy as restricted builds.)
Impartial, *Undirected*, Restricted Builds, *Max Height* 4

Can simulate *GEOGRAPHY*

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Game Complexity: Between Geography and Santorini
Impartial, *Undirected*, Restricted Builds, *Max Height 4*

- “Hard!”
- Can simulate GEOGRAPHY
The Game “20133”

- Board: A path with 5 vertices.
- Initial heights: 2, 0, 1, 3, 3.
- Max height 4.
- Worker starts on the 2.
Properties of “20133”

- Bipartite graph → Worker position determines whose turn. (Helps make analysis tractable.)

First player loses, but can force reaching the rightmost vertex. (Play will always progress from one meta-vertex to another.)

Once this happens, if game were to “restart” on the resulting board, first player would win. (Leaving a meta-vertex “deletes” it.)

First player would win if worker started on the rightmost vertex, and leftmost vertex cannot be reached. (Moving backwards among “meta-vertices” is punished.)
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\[20133(A) \rightarrow 20134(B) \rightarrow 20144(A) \rightarrow 20244(B)\]

A wins
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Found via computer search.
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Found via computer search. Search failed for max height 3 and for unrestricted builds.
"Hard"

\[ \Phi = (x_1 \lor x_2 \lor x_3) \land (\overline{x_1} \lor \overline{x_2} \lor x_{2n}) \land \cdots \]
“Hard” (Partizan GEOGRAPHY) (Fraenkel, Simonson 1993)

\[ \Phi = (x_1 \lor \overline{x_2} \lor x_3) \land (\overline{x_1} \lor \overline{x_2} \lor x_{2n}) \land \cdots \]

This is a new reduction.
Partizan, Directed, Unrestricted Builds

“Hard”

N - 1 Chain of Clause Deleters

N - 1 Length Escape Chain

P2 Start

X₂

not X₂

X₄

not X₄

P2 Start

X₁

not X₁

X₃

not X₃

N Clause Nodes (each clause node points to variable chains for each of its variables)

N - 1 Length Delay Chain

N - 2 Length Variable Chain (with doubled end nodes)

Image Credit: Carson Geissler

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Partizan, Unirected

- Restricted Builds (Undirected Partizan GEOGRAPHY)
  - Previously: NP-Hard (Fraenkel, Scheinerman, Ullman 1993)
  - Preliminary Result: PSPACE-Hard
- Unrestricted Builds: PSPACE-Hard (Preliminary)
Open Problems

- Impartial, Undirected, Restricted Builds, Max Height 3
- Impartial, Undirected, Unrestricted Builds, Max Height $\geq 3$
- Grids
- Tower wins
- Questions about worker placement
- Adding workers to “easy” games
- Restrictions to “obtainable” positions
- Parametric dependence on max height
- Varied movement restrictions
- Other building restrictions