Game Complexity: Between Geography and Santorini

Nathan Fox

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Joint work with Carson Geissler
Santorini


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Santorini

Adjustable Parameters

Underlying Board
- Grid or general graph?
- Directed or undirected?

Game Rules
- “Tower win” rule?
- Number of workers?
- Only build where you came from?
- Restriction on moving upward?
- Shared workers?
“Easy”: Can describe a winning strategy concisely.  
(Formally: In P.)

“Hard”: Known or not believed to have a concise winning strategy.  
(Formally: PSPACE-hard.)
Comparing Complexities: Reductions

Game $A$ is at least as hard as Game $B$ if you can simulate $B$ using the rules of $A$. 
Starting Point: TQBF Game

Setup: $3$-CNF Boolean formula $\Phi$ with variables $x_1, x_2, \ldots, x_{2n}$

Rules: Players take turns setting variables in order $x_1, x_2, \ldots, x_{2n}$.

End of Game: First player wins iff $\Phi$ is true.

Classical "hard" game
Basic Santorini Reductions

- General graphs are at least as hard as grids.
- Directed graphs are at least as hard as undirected graphs.
- Games with additional workers are at least as hard as games without.
- Higher max heights are at least as hard as lower max heights.
Basic Santorini Reductions

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- Directed graphs are at least as hard as undirected graphs.
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- Higher max heights are at least as hard as lower max heights.
- Max height 1 is at least as hard as max height 2 (without “tower wins”)

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Game Complexity: Between Geography and Santorini
Our Assumptions Going Forward

- General graphs (directed or undirected)
- No “tower wins”
- 1 worker (shared), or 1 worker per player (impartial vs. partizan)
- Max height 1 unless otherwise stated
- Same upward movement restrictions as Santorini
- “Restricted builds” means build only where moved from
Impartial, Directed, Restricted Builds

Classically "hard" (Schaefer 1978; Lichtenstein, Sipser 1980)

Φ = (x₁ ∨ x₂ ∨ x₃) ∧ (x₁ ∨ x₂ ∨ xₙ) ∧ ···
Impartial, Directed, Restricted Builds

- a.k.a. GEOGRAPHY
- Classically “hard” (Schaefer 1978; Lichtenstein, Sipser 1980)
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\[ \Phi = (x_1 \lor \overline{x_2} \lor x_3) \land (\overline{x_1} \lor \overline{x_2} \lor x_{2n}) \land \ldots \]
Impartial, *Undirected*, Restricted Builds

- a.k.a. Undirected GEOGRAPHY
Impartial, *Undirected*, Restricted Builds

- a.k.a. Undirected GEOGRAPHY
- “Easy!” (Fraenkel, Scheinerman, Ullman 1993)
- Strategy: If vertex with worker is involved in every maximum-size matching, choose one and win by moving along matching edges.
Impartial, Undirected, Unrestricted Builds

- "Easy"
- Strategy: If vertex with worker is involved in every maximum-size matching, choose one and win by moving along matching edges and building where moved from. (Same strategy as restricted builds.)
Impartial, *Undirected*, Restricted Builds, *Max Height* 4
Impartial, *Undirected*, Restricted Builds, *Max Height* 4

- “Hard!”
- Can simulate GEOGRAPHY
The Game “20133”

- Board: A path with 5 vertices.
- Initial heights: 2, 0, 1, 3, 3.
- Max height 4.
- Worker starts on the 2.
Properties of “20133”

- Bipartite graph $\rightarrow$ Worker position determines whose turn. (Helps make analysis tractable.)

First player loses, but can force reaching the rightmost vertex. (Play will always progress from one meta-vertex to another.) Once this happens, if game were to “restart” on the resulting board, first player would win. (Leaving a meta-vertex “deletes” it.) First player would win if worker started on the rightmost vertex, and leftmost vertex cannot be reached. (Moving backwards among “meta-vertices” is punished.)

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\[
\begin{align*}
34 \ (B) & \rightarrow A \ 	ext{wins} \\
334 \ (B) & \rightarrow 334 \ (A) \rightarrow 334 \ (B) \rightarrow A \ 	ext{wins} \\
33234 \ (B) & \rightarrow 3234 \ (A) \rightarrow 3234 \ (B) \rightarrow A \ 	ext{wins} \\
3234 \ (B) & \rightarrow 3234 \ (A) \rightarrow 3234 \ (B) \rightarrow A \ 	ext{wins} \\
A \ 	ext{wins} & \rightarrow 3234 \ (B)
\end{align*}
\]
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\[
\text{20133(A)} \rightarrow \text{20134(B)} \rightarrow \text{20144(A)} \rightarrow \text{20244(B)} \\
A \text{ wins}
\]
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Found via computer search.
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Found via computer search.
Search failed for max height 3 and for unrestricted builds.
Impartial, Directed, *Unrestricted Builds*

“Hard”

\[ \Phi = (x_1 \lor x_2 \lor x_3) \land (\overline{x}_1 \lor \overline{x}_2 \lor x_{2n}) \land \cdots \]
“Hard” (Partizan GEOGRAPHY) (Fraenkel, Simonson 1993)

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This is a new reduction.
Partizan, Directed, *Unrestricted Builds*

“Hard”

Image Credit: Carson Geissler
Partizan, Undirected

- Restricted Builds (Undirected Partizan GEOGRAPHY)
  - Previously: NP-Hard (Fraenkel, Scheinerman, Ullman 1993)
  - Preliminary Result: PSPACE-Hard
- Unrestricted Builds: PSPACEACE-Hard (Preliminary)
Open Problems

- Impartial, Undirected, Restricted Builds, Max Height 3
- Impartial, Undirected, Unrestricted Builds, Max Height $\geq 3$
- Grids
- Tower wins
- Questions about worker placement
- Adding workers to “easy” games
- Restrictions to “obtainable” positions
- Parametric dependence on max height
- Varied movement restrictions
- Other building restrictions