The Talk

The game CUT for $\mathcal{C} = \{1, 2c\}$

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Impartial, Normal-Play Games

A combinatorial Game is a 2-player, sequential, loopfree game with perfect information.

Today we restrict our attention to games which are:

impartial – both players have access to the same options (e.g. NOT chess)

AND Normal-play – the player who makes the last legal move wins.

($\approx$ if it is your turn, and you have no legal moves, you lose)

$\boxed{\text{Example – Take Away 1,2,3}}$

Setup There is a pile of tokens on the table.

Options You can remove 1, 2, or 3 tokens from the pile

{ A position is a winning position if the player whose turn it is can guarantee a win
{ A position is a losing position if the other player can

In this example, we can classify all positions with a simple chart:

	n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
win/Loss?		L	W	W	W	L	W	W	W	L	. .							
NIM-Value	G(n)	0	1	2	3	0	1	2	3	0	. .							

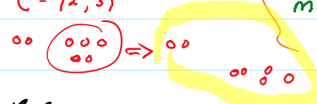
Next, we recursively define the NIM VALUE $G(n)$ of a position n as the MEX of the values of its options. <Compute the NIM VALUES for the above example>

Minimal
Excluded
Value
MEV ??

$\text{mex}\{0,3,3,1,5\} = 2$

Our Game: CUT

$\mathcal{C} = \{2,3\}$



Setup: There are finitely-many finite piles on the table.

Options: Each version of CUT is specified by a cut-set $\mathcal{C} \subseteq \mathbb{N}$.

Then an option is to select one of the piles and some $d \in \mathcal{C}$, and replace the pile with a partition with $d+1$ parts

NB Total # of tokens in play does not change!

But then how do we calculate Nim-values for a position with multiple piles?

The Sprague-Grundy Theorem (1936) says that, since CUT is disjunctive [each move only affects a single pile], we add the values of the piles together using the NIM-SUM, which is the binary sum ~~modulo~~ 2.

NB $a \oplus a = 0$

Ex $9 \oplus 11 = 1001_2 \oplus 1011_2 = 0010_2 = 2$

COR We only care about finding NIM-values of single pile positions

Warm-up: Some Examples of CUT from Dailly, Duchene, Larsson, Paris (2020)

CUT was defined pretty darn recently...

Ex 1 (Prop 3) Suppose $1 \in \mathcal{C}$ and $\mathcal{C}$ only contains odd numbers. ($\mathcal{C}$ can be infinite)

n	~~0~~	1	2	3	4	5	6	. . .

Ex 1 (Prop 3) Suppose $1 \in \mathcal{C}$ and $\mathcal{C}$ only contains odd numbers. ($\mathcal{C}$ can be infinite)

n	~~0~~	1	2	3	4	5	6	...
$G(n)$	~~0~~	0	1	0	1	0	1	...

Ex 2 (Prop 7) Suppose $\{1,2,3\} \subseteq \mathcal{C}$. Then $G(n) = n-1$.

Proof Sketch: For a given n, for a given $g < n-1$, we must find an option O_n with $\mathcal{G}(O_n) = g$.

Here's How:

O_n	n odd	n even
g odd	$(g+1, 1, \frac{n-g-2}{2}, \frac{n-g-2}{2})$	$(g+1, \frac{n-g-1}{2}, \frac{n-g-1}{2})$
g even	$(g+1, \frac{n-g-2}{2}, \frac{n-g-2}{2})$	$(g+1, 1, \frac{n-g-1}{2}, \frac{n-g-1}{2})$

NB: $a \oplus a = 0$

⟨what we left out: prove there is no option of n with NIM-value of $n-1$⟩

Ex 3 (Prop 8). Let $\mathcal{C} = \{1, 3, 2k\}$ for some $k \in \mathbb{N}$.

Then the NIM-sequence is $(0,1)^k(+2)$ ⟵ Arithmetic-Periodic Notation

The $(+2)$ is called the saltus ⟶ "jump" in Latin

OR
$$\begin{cases} 0,1,0,1,\dots,0,1 \\ 2,3,2,3,\dots,2,3, \\ 4,5,4,5,\dots,4,5,\dots \end{cases}$$

DDLP proved that for $c = 2,3,4,5$, the NIM-sequence for CUT when $\mathcal{C} = \{1, 2c\}$ is

$$(0,1)^c, (2,3)^c, 1, 4, (5,4)^{c-1}, (3,2)^c, (4,5)^c, (6,7)^c (+8)$$

-OR-

$$\left.\begin{array}{l} 0,1,0,1,\dots,0,1, \\ 2,3,2,3,\dots,2,3, \\ 1,4,5,4,\dots,5,4, \\ 3,2,3,2,\dots,3,2, \\ 4,5,4,5,\dots,4,5, \\ 6,7,6,7,\dots,6,7,\dots (+8) \end{array}\right\}$$

This is just the first period.
It has length $12c$.
It has 6 "rows", each of length $2c$

Theorem (E-Thanatipanonda, 2022) This is true for any $c \geq 2$.

Unfortunately, the proof is long,
but I hope to show you Aek's insight
about analyzing NIM-sets which otherwise seem irrelevant.

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Definition The NIM-set $\mathcal{N}(n, p, \mathcal{C})$ is set of NIM-values that results from splitting n tokens into p piles. The NIM-values are then calculated according to $\mathcal{C}$.

The first 17 nim-sets of CUT for $\mathcal{C} = \{1, 6\}$.

n	1	2	3	4	5	6	7	8	9	10
$\mathcal{N}(n, 2, \{1,6\})$	–	{0}	{1}	{0}	{1}	{0}	{1}	{0,2}	{1,3}	{0,2}
$\mathcal{N}(n, 7, \{1,6\})$	–	–	–	–	–	–	{0}	{1}	{0}	{1}
$\mathcal{G}_{\{1,6\}}(n)$	0	1	0	1	0	1	2	3	2	3

n	11	12	13	14	15	16	17
$\mathcal{N}(n, 2, \{1,6\})$	{1,3}	{0,2}	{3}	{0,1,2}	{0,1,3,4}	{0,1,2,5}	{0,1,3,4}

The first 17 nim-sets of CUT for $\mathcal{C} = \{1, 6\}$.

n	1	2	3	4	5	6	7	8	9	10
$\mathcal{N}(n, 2, \{1,6\})$	–	{0}	{1}	{0}	{1}	{0}	{1}	{0,2}	{1,3}	{0,2}
$\mathcal{N}(n, 7, \{1,6\})$	–	–	–	–	–	–	{0}	{1}	{0}	{1}
$\mathcal{G}_{\{1,6\}}(n)$	0	1	0	1	0	1	2	3	2	3

n	11	12	13	14	15	16	17
$\mathcal{N}(n, 2, \{1,6\})$	{1,3}	{0,2}	{3}	{0,1,2}	{0,1,3,4}	{0,1,2,5}	{0,1,3,4}
$\mathcal{N}(n, 7, \{1,6\})$	{0}	{1}	{0,2}	{1,3}	{0,2}	{1,3}	{0,2}
$\mathcal{G}_{\{1,6\}}(n)$	2	3	1	4	5	4	5

So Aek got out his Maple and started computing...

After all, if the NIM-sequences for, let's say, $\mathcal{C} = \{1, 8\}$ is going to be a "shift-expanded" version of the NIM-sequence for $\mathcal{C} = \{1,6\}$, then there is probably a correspondence between their NIM-sets, right?

Yes, but not how you might expect. In fact, it is probably more than you expect...

For example, the initial values of the sequence $\mathcal{N}(n, 3, \{1, 6\})$ are

{}, {},
{0}, {1}, **{0}**, {1}, {0}, {1},
{0, 2}, {1, 3}, **{0, 2}**, **{1, 3}**, {0, 2}, {1, 3},
{0, 1, 2}, {0, 1, 3, 4}, **{0, 1, 2, 5}**, **{0, 1, 3, 4}**, {0, 1, 2, 5}, {0, 1, 3, 4},

and the initial values of the sequence $\mathcal{N}(n, 3, \{1, 8\})$ are

{}, {},
{0}, {1}, **{0}**, **{1}**, **{0}**, **{1}**, {0}, {1},
{0, 2}, {1, 3}, **{0, 2}**, **{1, 3}**, **{0, 2}**, **{1, 3}**, {0, 2}, {1, 3},
{0, 1, 2}, {0, 1, 3, 4}, **{0, 1, 2, 5}**, **{0, 1, 3, 4}**, **{0, 1, 2, 5}**, **{0, 1, 3, 4}**{0, 1, 2, 5}, {0, 1, 3, 4}

But $p = 3$ corresponds to $2 \in \mathcal{C}$, which is in the ruleset of Neither game!

In fact, this correspondence holds for all $c \geq 3$, and all $p \geq 2$!

Stranger yet, a key ingredient in the proof of this is:

LEMMA

$$\boxed{\text{For all } p \geq 4,\ c \geq 2 \qquad \mathcal{N}(n+1, p+1, \{1, 2c\}) = \mathcal{N}(n, p, \{1, 2c\})}$$

again, most values of p seem irrelevant to the game in question...

That's probably all we have time for, but the main theorem, together with this lemma do give us this nice generalization!

COR Let $\mathcal{C}_1 = \{1, 2c_1, 2c_2, \ldots\}$ where $2 \leq c_1 \leq c_2 \leq \ldots$

$\mathcal{C}_2 = \{1, 2c_1\}$

Then $G_{\mathcal{C}_1}(n) = G_{\mathcal{C}_2}(n)$ for all n.

pf

(*) If $p \geq 4$, $c \geq 2$,

$$\mathcal{N}(n, p+2, \{1, 2c\}) = \mathcal{N}(n-2, p, \{1, 2c\}) \quad \text{Apply the Lemma twice}$$
$$\subseteq \mathcal{N}(n, p, \{1, 2c\}) \quad \text{Exercise}$$

But then
$$G_{\mathcal{C}_1}(n) = \operatorname{mex}\left\{ \mathcal{N}(n, 2, \mathcal{C}_1) \cup \left[\bigcup_{2c_i \in \mathcal{C}_1} \mathcal{N}(n, 2c+1, \mathcal{C}_1)\right]\right\}$$
$$= \operatorname{mex}\left\{ \mathcal{N}(n, 2, \mathcal{C}_2) \cup \left[\bigcup_{2c_i \in \mathcal{C}_1} \mathcal{N}(n, 2c+1, \mathcal{C}_2)\right]\right\} \quad \text{by induction}$$
$$= \operatorname{mex}\left\{ \mathcal{N}(n, 2, \mathcal{C}_2) \cup \mathcal{N}(n, 2c_1+1, \mathcal{C}_2)\right\} \quad \text{by (*)}$$

$$= G_{e_2}(n) \quad \square$$

Open Questions + Bibliography on the next pages $\longrightarrow$

Finding Patterns – Key observations

Observation (DDLP) Since the saltus is a power of 2, greater than each NIM-value in the first period, much of our analysis will only need to be done in this first period, more or less.

Decompositions

0, 1, 0, 1, . . . , 0, 1,
2, 3, 2, 3, . . . , 2, 3,
1, 4, 5, 4, . . . , 5, 4,
3, 2, 3, 2, . . . , 3, 2,
4, 5, 4, 5, . . . , 4, 5,
6, 7, 6, 7, . . . , 6, 7, .

Generally True:

$$G(n) \oplus G(n) \oplus G(m+1) \oplus G(m+1) = 0 = G(n+1) \oplus G(n+1) \oplus G(m) \oplus G(m)$$

True in the first period (shown):

$$G(2c) \oplus G(12c) = G(4c) \oplus G(10c) = G(6c) \oplus G(8c) = 6$$

True in this chart, but not just in the 1st period:

$G(n+1) = G(n) \oplus 1$ if $n \not\equiv 0 \bmod 2c$ and $n \not\equiv 4c+1 \bmod 12c$ (The rows are $\oplus 1$ – Alternating)

$$G(n) \oplus G(n) \oplus G(2ac) = G(n+1) \oplus G(n+1) \oplus G(2ac-2)$$

$$G(n+1) \oplus G(n+1) \oplus G(2ac+1) = G(n) \oplus G(n) \oplus G(2ac+3) \quad \text{if } a \not\equiv 2 \bmod 6$$

$$G(n+1) \oplus G(n+1) \oplus G(2ac+2) = G(n) \oplus G(n) \oplus G(2ac+4) \quad \text{if } a \equiv 2 \bmod 6$$

The first 30 nim-sets of CUT for $\mathcal{C} = \{1, 6\}$, $p = 2$, decomposed into alternating subsequences.

n	1	2	3	4	5	6	7	8	9	10
$\mathcal{N}(n, 2, \{1, 6\})$	–	0	1	0	1	0	1	0 2	1 3	0 2

n	11	12	13	14	15	16	17	18	19
$\mathcal{N}(n, 2, \{1, 6\})$	1 3	0 2	3	2 1 0	3 0 1 4	2 1 0 5	3 0 1 4	2 1 0 5	0 1 4

n	20	21	22	23	24	25	26	27	28	29	30
$\mathcal{N}(n, 2, \{1, 6\})$	0 5 3	1 4 2 6	0 5 3 7	1 4 2 6	0 5 3 7	2 6	3 7 0 1 4	2 6 0 5	3 7 1 4 0	2 6 0 5 1	3 7 1 4 0

The first subsequence begins at $O_2 = (1, 1)$ and ends at $O_{12} = (6, 6)$

The second subsequence begins at $O_8 = (7, 1)$ and ends at $O_{18} = (12, 6)$
The third subsequence begins at $O_{14} = (13, 1)$ and ends at $O_{19} = (13, 6)$
The fourth subsequence begins at $O_{14} = (7, 7)$ and ends at $O_{24} = (12, 12)$
The fifth subsequence begins at $O_{15} = (14, 1)$ and ends at $O_{24} = (18, 6)$
The sixth subsequence begins at $O_{20} = (13, 7)$ or $O_{20} = (19, 1)$ and ends at $O_{30} = (24, 6)$
The seventh subsequence begins at $O_{21} = (14, 7)$ and ends at $O_{30} = (18, 12)$
The eighth subsequence begins **and** ends at $O_{26} = (13, 13)$
The ninth subsequence begins at $O_{26} = (19, 7)$ ...
The tenth subsequence begins at $O_{26} = (25, 1)$ ...
The eleventh subsequence begins at $O_{28} = (14, 14)$ ...

Definition

(A) Call $1 \bmod 2c$ and $(4c+2) \bmod 12c$ "In numbers"

(B) Call $0 \bmod 2c$ and $(4c+1) \bmod 2c$ "out numbers"

(C) A partition is "entering/exiting" if all of its parts are in/out numbers

Prop (A) If O_n is a non-exiting option, then there is an option O_{n+1} s.t. $G(O_n) = G(O_{n+1}) \oplus 1$

(B) If O_n is a non-entering option, then there is an option O_{n-1} s.t. $G(O_n) = G(O_{n-1}) \oplus 1$

Prop Let $p \geq 4$. If an entering partition in $N(n, p, \mathcal{C})$ has no part of size 1, then it has the the same NIM-value as a non-entering partition in $N(n, p, \mathcal{C})$

Prop Let $p \geq 4$. Then every exiting partition in $\mathcal{N}(n, p, \mathcal{C})$ has the same NIM-value as a non-exiting partition in $\mathcal{N}(n, p, \mathcal{C})$.

Open Qs

Some known results about CUT.

Cut-set $\mathcal{C}$	Nim sequence	Proposition in [DDLP, 2020
$1 \notin \mathcal{C}$	$(0)^c(+1)$ where c is the smallest element in $\mathcal{C}$	6
$1 \in \mathcal{C}$ and $\mathcal{C}$ contains only odd numbers	$(0, 1)$	3
$\{1, 2, 3\} \subseteq \mathcal{C}$	$(0)(+1)$ (i.e. $\mathcal{G}(n) = n - 1$)	7
$\mathcal{C} = \{1, 3, 2c\}$	$(0, 1)^c(+2)$	8

(E-THANATIPANONDA, 2022)

[redacted]. *The nim-sequence of the game* CUT *with cut-set* $\mathcal{C} = \{1, 2c\}$ *for any* $c \geq 2$ *is precisely*

$$(0, 1)^c(2, 3)^c, 1, 4, (5, 4)^{c-1}, (3, 2)^c(4, 5)^c(6, 7)^c(+8).$$

OR even $\{1, 2c_1, 2c_2, \ldots\}$ where $2 \leq c_1 < c_2 < \ldots$

In this section, we categorize the families of cut sets for which the nim-sequence of CUT remains unknown. There are 4 such families. Let X to be a non-empty set of even numbers, each of which is at least 4. Let Y to be a non-empty set of odd numbers, each of which is at least 5. Let $x = 2c$ and y be the smallest elements of X and Y, respectively.

Family A: $\mathcal{C} = \{1, 3\} \cup X$, or $\mathcal{C} = \{1, 3\} \cup X \cup Y$

We already know from [3, Propositions 8] that the nim-sequence for $\mathcal{C} = \{1, 3, 2c\}$ is $(0, 1)^c(+2)$.

Conjecture 1. *The nim-sequence for all games of* CUT *in this family are all precisely* $(0, 1)^c(+2)$.

Family B: $\mathcal{C} = \{1\} \cup X \cup Y$

The nim-sequence of this family seems to have some resemblance to the nim-sequence for $\mathcal{C} = \{1, 2c\}$ when $c \geq 2$, but we cannot make a full conjecture at this time. The following partial extension of Theorem 19 seems to be true.

Conjecture 2. *If* $3x < y$, *then* $\mathcal{G}_{\mathcal{C}}(n) = \mathcal{G}_{\{1,x\}}(n)$ *for* $n \geq 1$.

We note that proving the arithmetic-periodicity of Families A and B would imply Conjecture 1 of [3].

Family C: $\{1, 2\} \subseteq \mathcal{C}, 3 \notin \mathcal{C}, \mathcal{C} \neq \{1, 2\}$.

It is not so clear how to categorize the patterns of this family. However, we do observe:

Conjecture 3. *The nim-sequence for all games of* CUT *in Family C are all* ultimately *arithmetic-periodic.*

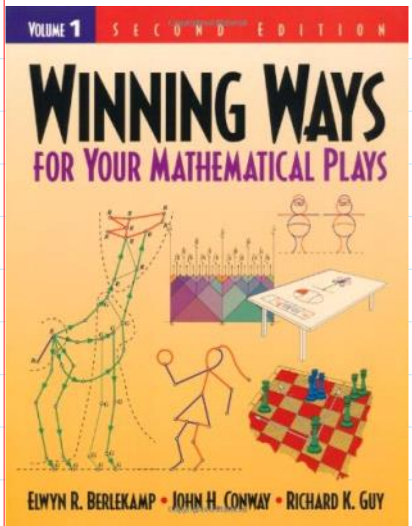
Family D: $\mathcal{C} = \{1, 2\}$

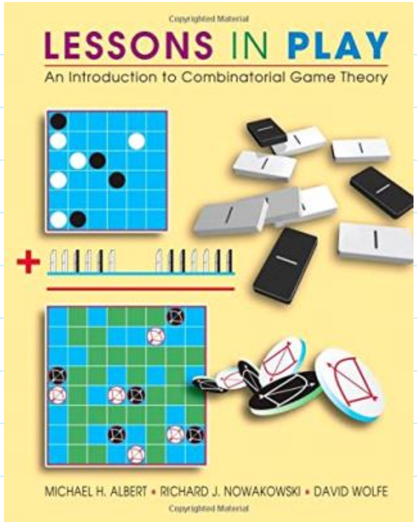
The first 36 terms of the nim-sequence for this game of CUT are

0,	1,	2,	3,	1,	4,	3,	2,	4,	5,	6,	7,
8,	9,	7,	6,	9,	8,	11,	10,	12,	13,	10,	11,
13,	12,	15,	14,	16,	17,	5,	15,	17,	16,	19,	18

It is supposed that the nim-sequence for this particular version of CUT is the most difficult to analyze. We also cannot find any pattern here. In [3], it was shown that this game is equivalent to the take-and-break game with hexadecimal code 0.7F.

Bibliography





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Partition games

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The arithmetic-periodicity of CUT for $\mathscr{C} = \{1, 2c\}$

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These Slides: https://tinyurl.com/54ddp53a

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