GAMBLER'S RUIN WITH k-GAMBLER'S

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1. THE BIG PICTURE

$\mathcal{X}$ - FINITE SET

$K(x,y) \geq 0$, $\sum_y K(x,y) = 1$ MARKOV CHAIN

$B \subseteq \mathcal{X}$ SUBSET

PROBLEM START AT $x_0 \in B$, RUN THE MARKOV CHAIN UNTIL YOU EXIT B (WHEN YOU DIE).

- LET T BE FIRST EXIT TIME

? WHATS $P_{x_0}(T > t)$ HOW LONG TO EXIT?

? WHEN YOU EXIT, WHERE ARE YOU?

$$P_{x_0}\{X_T = y\}$$

AND

? HOW DO THE ANSWERS DEPEND ON x_0 ?

2. EXAMPLE GAMBLERS RUIN WITH TWO GAMBLERS

$\mathcal{X} = \{0, 1, \ldots, N\}$, $K(i,i+1) = K(i,i-1) = K(0,0) = K(N,N) = \frac{1}{2}$

$B = \{1, 2, \ldots, N-1\}$, STOP WHEN HIT 0 OR N

$E_{x_0}(T) = x_0(N+1-x_0)$

$P_{x_0}(X_T = N) = \frac{x_0}{N+1}$, $P_{x_0}(X_T = 0) = 1 - \frac{x_0}{N+1}$

3. EXAMPLE 1 GAMBLERS RUIN WITH 3 GAMBLERS

3 GAMBLERS START WITH A, B, C

- EACH STEP, PICK 2 (UNIFORMLY) FLIP A FAIR COIN AND TRANSFER 1

- EVENTUALLY (TIME T_1) ONE GAMBLER IS ELIMINATED AND THEN ORDINARY GAMBLERS RUIN RUNS WITH THE LAST TWO

- T_2 TIME GAME ENDS

BACHLIER (1912):

		100, 100, 100	1, 1, 298
$E(T_1) =$	$\frac{3ABC}{A+B+C}$	10,000	2.98
$E(T_2) =$	$AB + AC + BC$	30,000	597

DISTRIBUTION OF X_{T_1}. HARDER L(COMING)

BUT CHANCE A WINS ALL $\frac{A}{A+B+C}$

Ex (3 GAMBLERS)

FROM 1,1,N-2; WHAT'S CHANCE C IS ELIMINATED FIRST?

Heuristic I $\quad ?\ \dfrac{c}{N}\ ?$

Heuristic ICM $\quad \dfrac{c}{N^2}$

Th $\quad \dot{=}\ \dfrac{c}{N^3} \qquad c = \dfrac{\sqrt{\pi}}{3\sqrt{3}}\left(\dfrac{\Gamma(\frac{1}{3})}{\Gamma(7/6)}\right)^3$

$\doteq 4.5597944996\ldots$

TABLE 4

The exact values of $P_{1,1,N-2}(321)$, rounded to 15 significant digits, suggesting that this quantity is asymptotic to c/N^3 for $c \doteq 4.5597945$

N	$P_{1,1,N-2}(321)$	$N^3 P_{1,1,N-2}(321)$
50	0.0000364783779008280	4.55979723760
100	0.00000455979467170448	4.55979467170
150	0.00000135105023226911	4.55979453391
200	0.000000569974313837992	4.55979451070
250	0.000000291826848279112	4.55979450436
300	0.000000168881277854908	4.55979450208

MARKOV CHAIN TOOLS (GENERAL $K(x,y)$), $B \subseteq \mathcal{X}$

$$\begin{array}{c|cc} & \text{BDRY} & \text{INT} \\ \hline \text{BDRY} & I & 0 \\ \text{INTERIOR} & S & Q \end{array}$$

<u>Th</u> FOR $x \in$ INT, $y \in$ BDRY,

$$P(x,y) = P_x(\text{CHAIN FIRST REACHES BDRY AT } y)$$

$$= (I-Q)^{-1} S$$

POISSON KERNEL, HARMONIC MEASURE

☹ FOR GAMBLER'S RUIN $(I-Q)^{-1}$ REQUIRES INVERTING AN $\binom{n-1}{2} \times \binom{n-1}{2}$ ($19{,}701 \times 19{,}701$ $N=200$)

NAIVE DOUBLE PRECISION TOOK ABOUT 97 HRS

MANY OTHER WAYS TO GET NUMBERS

DIACONIS P., ETHIER S. (2022) 'GAMBLER'S RUIN AND THE ICM'

5. FOR 3 PLAYER GAMBLER'S RUIN

DIACONIS, P., HOUSTON-EDWARDS, K., SALOFF-COSTE, L. (2021)

TREAT AS RANDOM WALK ON TRIANGLE $A+B+C=N$

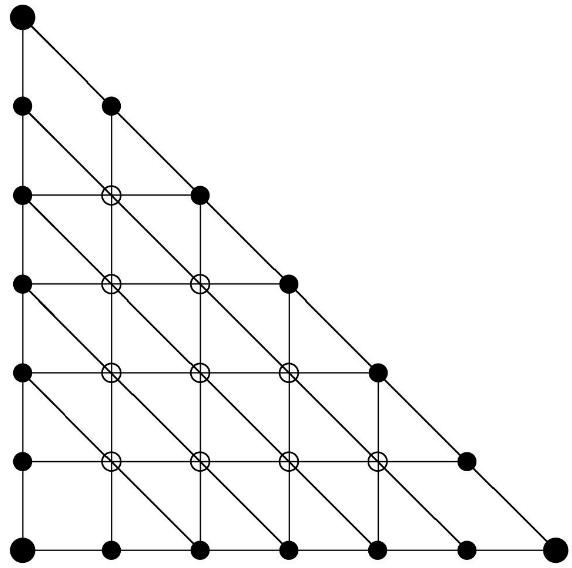

FIG. 1. *When $N = 6$, the state space $\mathscr{X}$ is represented by 28 dots, of which 10 are interior states (open dots), 15 are nonabsorbing boundary states (solid dots), and 3 are absorbing states (larger solid dots). Line segments show possible transitions. There are six from each interior state and two from each nonabsorbing boundary state.*

th $P_{x_1,x_2}(y,0)$

$$\asymp \frac{x_1 x_2 (x_1+x_2)(N-x_1-x_2)(N-x_2)\, y^2 (N-y)^2}{N^4 (x_1+d)^2 (x_2+d)^2 (x_1+x_2+2d)^2}$$

d = GRAPH DISTANCE FROM x_1, x_2 TO $y, 0$

$$a_N \asymp b_N$$

$\exists\ c, c' > 0$

$$c a_N \le b_N \le c' a_N$$

RESULT IS UNIFORM IN x_1, x_2, y

CORR $P_{1,1,N-2}$(PLAYER 3 GOES BROKE FIRST) $\asymp \frac{1}{N^3}$

6. A SUCCESS STORY FOR EXPERIMENTAL MATHEMATICS

RANDOM WALK APPROXIMATED BY BROWNIAN MOTION

ASSUMING THIS IS 'GOOD' SANDRA LEE

USED CONFORMAL MAPPING TECHNIQUES TO SOLVE

THE FIRST HITTING PROBLEM FOR B.M. ON TRIANGLE

AND FOUND THE MAGIC CONSTANT ABOVE

BUT ☹ B.M. APPROXIMATION IS NOT GOOD

TO $1/N^3$!!!

DENISOV, D. AND WACHTEL, V. (2022) 'HARMONIC MEASURE IN A MULTI-DIMENSIONAL GAMBLER'S PROBLEM'

MANAGED TO PROVE C/N^3 FOR THIS C

7. WHAT ABOUT 4 GAMBLERS

$$P_{1N-1}(\text{BIG PLAYER OUT FIRST}) \sim \frac{1}{N}$$

$$P_{11N-2}(\text{BIG PLAYER OUT FIRST}) \sim \frac{C}{N^3}$$

WHAT ABOUT P_{111N-3}(BIG PLAYER OUT FIRST) ?

HMM (O'CONNOR, SAKOFF-COSTE '4 PLAYER GAMBLER'S RUIN)

ITS $\approx 1/N^{\alpha}$, $\alpha = 5.68\ldots$

8. SOME (NICE) OPEN PROBLEMS WITH K GAMBLERS
EACH TIME, CHOOSE TWO AT RANDOM

- ALL IN: IF THEY HAVE A, B, BET min(A,B) + FLIP FAIR COIN (PLAYER WITH SMALLER CHIP COUNT IS ELIMINATED OR DOUBLES UP).

- OCCASIONALLY ALL IN: BET SIZE UNIFORM ON 1, 2, ..., min(A,B)

- COMPULSIVE GAMBLER: WITH TWO PLAYERS, ONE GETS OTHER'S MONEY WITH PROBABILITY $\frac{A}{A+B}$ OR $\frac{B}{A+B}$

4. BACK TO GENERAL SET UP $\mathcal{X}$, $K(x,y)$, B

$\exists$ QUASI-STATIONARY DISTRIBUTION, $x \in B$

$$\pi(x) = \lim_{t\to\infty} P(X_t = x \mid T > t)$$

? WHAT IS THIS ?

AND

? FIND RATES OF CONVERGENCE ?

$$\| P_x^t - \pi \|_{TV}$$

SEE DIACONIS, P., HOUSTON-EDWARDS, K., SALOFF-COSTE (2022) 'ANALYTIC-GEOMETRIC METHODS FOR FINITE MARKOV CHAINS WITH APPLICATIONS TO QUASI-STATIONARITY' ALEA.