Voting on Cyclic Orders, Representations, and Ties

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Gordon College

Rutgers Experimental Mathematics Seminar
April 28, 2022

Outline

Voting Theory

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Question

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In both of these cases, note that we really just consider how many people vote for each ballot.

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What is voting theory?

Principle

Let B be a set of ballots and O be a set of outcomes. We can think of (anonymous) voting as functions from the set of *profiles* on B (elements of $\mathbb{Q}^{|B|}$) to (the power set of) O.

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Example

If $\mathcal{A}$ is the set of all programming languages, let B be the set of ballots ranking your top five favorites, and O be the set of full linear orders $\mathcal{L}(\mathcal{A})$. (Any procedure.)

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Our goal in this talk is as follows:

- Briefly explain how representation theory connects to voting.
- Introduce a novel outcome, *cyclic orders*, and some ballots/procedures related to this.
- Discuss experimental work done by my students on this, as well as theoretical results.

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For a given O there could be different ballots B, and vice versa.

Linear Voting

Famous Example

Let $\mathcal{A}$ be the set of candidates for Holy Roman Emperor, and B be the set of full rankings $\mathcal{L}(\mathcal{A})$. Assign one point for the candidate at the bottom of a voter's ranking, two points for the next one, up through $n = |\mathcal{A}|$ points for their top-ranked candidate. Then take the argmax of the summation outcome vector.

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If we order ballots for $\mathcal{A} = \{A, B, C\}$ as $A \succ B \succ C$, $A \succ C \succ B$, $C \succ A \succ B$, $C \succ B \succ A$, $B \succ C \succ A$, $B \succ A \succ C$, then the following matrix assigns points in this system.

$$M = \begin{pmatrix} 3 & 3 & 2 & 1 & 1 & 2 \\ 2 & 1 & 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 3 & 2 & 1 \end{pmatrix}$$

With $\mathbf{p} = (2, 1, 0, 0, 0, 1)^T$ we get $M\mathbf{p} = (10, 8, 5)^T$ and A is the winner.

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This method is due to Nicolas Cusanus (15th century) and J.-C. Borda (18th century), and is usually called the *Borda Count*. There are many other similar points-based systems – you may have used one yourself in voting in a professional society, for Parliament if you are from Nauru or Slovenia, or for the Eurovision song contest.

Bringing in Groups

It should be clear that $\mathcal{A}$ has an action of the symmetry group S_n.

- Note that S_n can be applied, by extension, to both the set of ballots B as well as the outcomes O.
- Further, by linearity, we can let S_n act on the set of *profiles* as well as the *outcome vectors*.

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Even the matrix seems very symmetric.

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Let's use the permutation $\sigma = (AB)$, which changes $\mathbf{p} = (2, 1, 0, 0, 0, 1)^T$ to $\sigma\mathbf{p} = (1, 0, 0, 0, 1, 2)^T$. Then with M as before:

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we get $M\mathbf{p} = (8, 10, 5)^T$ and B is the winner.

It turns out that this M is invariant under S_3, in the sense that if we uniformly change the names of the candidates on everyone's ballot, the outcome is also changed by exactly this permutation.

Bringing in Groups

In the linear context, this group invariance is identical to a famous voting theory criterion.

Fact

Given a choice procedure F on profiles $\mathbf{p}$ on B, if

$$\sigma(F(\mathbf{p})) = F(\sigma(\mathbf{p})) \text{ for all } \sigma \text{ and } \mathbf{p} \,,$$

under the action of S_n, then we say the procedure is *neutral*.

Technical note: For a given profile or outcome, we must define $\sigma(\mathbf{p})[x] = \mathbf{p}[\sigma^{-1}x]$.

Representations and Voting

Now comes the real firepower. Summarizing the properties:

- The sets of profiles P and vote totals Q are acted on by S_n
- Our voting function F is a *linear* procedure (e.g. points-based voting rule) over $\mathbb{Q}$
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We can now invoke standard facts about representations.

Representations and Voting

Fact

Every (finitely generated) $\mathbb{Q}S_n$-module decomposes (in a computable way) as a direct sum of a finite set of *irreducible* $\mathbb{Q}S_n$-submodules. This decomposition is unique up to multiple copies of isomorphic submodules.

Fact

For any $\mathbb{Q}S_n$-module homomorphism $F : M \to N$ and an irreducible submodule $U \subseteq M$, either $F(U) = 0$, or $U \simeq F(U) \subseteq N$ is an isomorphism (in fact, multiplication by a constant).

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(The latter is essentially the famous *Schur's Lemma* with the note that $\mathbb{Q}$ is a splitting field for S_n.)

Representations and Voting

Suppose $B = \mathcal{L}(\mathcal{A})$ and $O = \mathcal{A}$. Then P is the $n!$-dimensional regular representation of S_n (over $\mathbb{Q}$) and $Q \simeq S^{(n)} \oplus S^{(n-1,1)}$ (a permutation representation).

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Example

For $n = 3$, we have $P \simeq \mathbb{Q}^6 \simeq S^{(3)} \oplus S^{(2,1)^{\oplus 2}} \oplus S^{(1,1,1)}$ and $Q \simeq \mathbb{Q}^3 \simeq S^{(3)} \oplus S^{(2,1)}$.

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Example, details

$S^{(3)}$ is just the space spanned by $(1,1,1,1,1,1)^T$.
A typical $S^{(2,1)}$ profile vector is one with one voter for each ranking with A in first place and 'negative one voters' for each ranking with A in last place; the outcome $S^{(2,1)}$ vectors are similar, such as two points for A and 'negative one points' for B and C.

Representations and Voting

If our function is from some huge representation to $Q \simeq S^{(n)} \oplus S^{(n-1,1)}$, then everything in the profile space that isn't one of these irreducible submodules must be killed.

Example, continued

For $n = 3$, we have $P = S^{(3)} \oplus {S^{(2,1)}}^{\oplus^2} \oplus S^{(1,1,1)}$, so any system will kill half of the $S^{(2,1)}$ and all of the $S^{(1,1,1)}$

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Among all such systems, the Borda Count keeps the most 'voting-relevant' $S^{(2,1)}$ piece (Daugherty et al. (2009); see also Crisman (2014) for a different interpretation).

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Gaming at the Table

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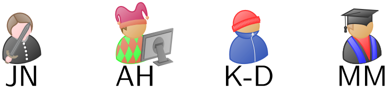

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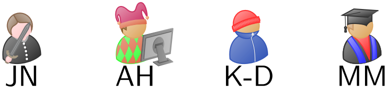

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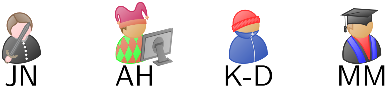
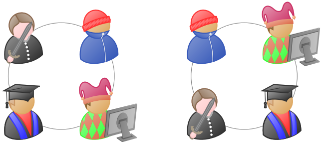


As with many combined games of skill and chance, the order of seating matters a great deal, especially if there are many rounds of play, with an equal number of times being the starting player.

Notice that it's only the *relative* order that matters, at least in the long run. These two orders are equivalent.



Cyclic Orders

Formally, we can say the following.

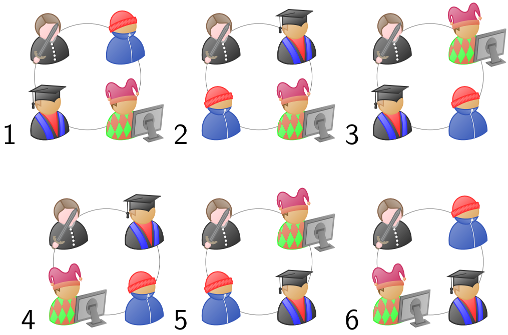
Definition

The set of permutations on a finite set $\mathcal{A} = \{A, B, C, \cdots\}$ has the equivalence relation where $A \succ B \succ C \succ D$ is identified with $B \succ C \succ D \succ A$, and so forth (under the (right) action of the cyclic group of order n).
The set of equivalence classes is called the set of *cyclic orders* on $\mathcal{A}$. If we have n candidates, we may call the set of cyclic orders CO_n.

There are $(n-1)!$ cyclic orders on a set with n elements, and we denote them for convenience as $ABCDA$ or $ABCD$, depending on context.

Gaming at the Table

How does voting connect? Suppose the spectators are making side bets and are allowed to *vote* on their favorite order!



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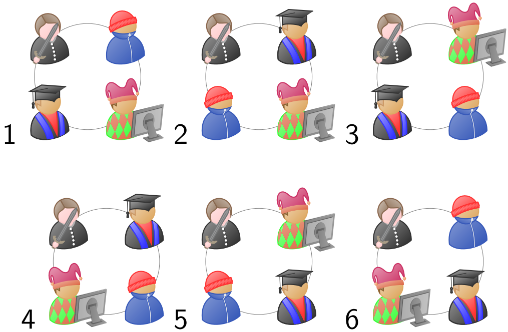
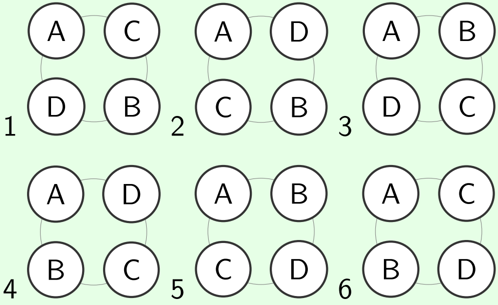


There are many similar settings where this could be useful!

Cyclic Orders

Cyclic Orders for $n = 4$

Here are the cyclic orders for $n = 4$, $\mathcal{A} = \{A, B, C, D\}$.



Note the ordering of the cyclic orders: $ACBD(A)$, $ADBC(A)$, $ABCD(A)$, $ADCB(A)$, $ABDC(A)$, $ACDB(A)$.

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The most obvious ballot is to ask each voter for just one cyclic order.
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- One point for the cyclic order on your ballot and *minus one* point for its reversal – if the direction *really* matters
- Two points for the cyclic order on your ballot and one point for its reversal – order and adjacency both matter

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- Organize the points into a matrix M.
- Let a profile be considered as a column vector, e.g. $\mathbf{p} = (2, 1, 0, 0, 0, 1)^T$.
  - (Recall the explicit order *ACBD*, *ADBC*, *ABCD*, *ADCB*, *ABDC*, *ACDB*.)
- Compute the vector $M\mathbf{p}$ for resulting point totals.

The final choice (or choices, if ties) is the argmax of the vector $M\mathbf{p}$.

Your ballot is a cyclic order

Example

Here is the matrix in question for the procedure, ‘Two points for the cyclic order on your ballot and one point for its reversal’:

$$M = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

With $\mathbf{p} = (2, 1, 0, 0, 0, 1)^T$ we get $M\mathbf{p} = (5, 4, 0, 0, 1, 2)^T$.

Points-based voting rules, again

This whole setting is nothing more than the previous points-based voting rules, but now with both $B = O = CO_n$ the set of cyclic orders! The matrix is a fancy way of organizing information such as, 'If you voted for $ABCD$, you give two points to $ABCD$ but one point to $ADCB$.'

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Further, there is once again a (left) group action on $B = O$.

Example ($n = 3$)

The permutation $\sigma = (ABC)$ leaves the cyclic order $ABCA$ invariant.

But (AB) applied to $ABCA$ would be $BACB = ACBA$, which is certainly different.

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Time to see what the representation theory tells us about the systems.

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Back to Cyclic Orders

Decomposition for $n = 4$

The space of profiles decomposes into $T \oplus U \oplus V$, where

- T has basis $\{(1,1,1,1,1,1)^T\}$
- U has basis $\{(1,-1,0,0,0,0)^T, (0,0,1,-1,0,0)^T, (0,0,0,0,1,-1)^T\}$,
- V is the (two-dimensional) span of $\{(2,2,-1,-1,-1,-1)^T, (-1,-1,2,2,-1,-1)^T, (-1,-1,-1,-1,2,2)^T\}$.

(For the cognoscenti: the decomposition is $S^{(4)} \oplus S^{(2,1,1)} \oplus S^{(2,2)}$.)

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(For the cognoscenti: the decomposition is $S^{(4)} \oplus S^{(2,1,1)} \oplus S^{(2,2)}$.)
Thus every neutral, points-based voting procedure on this space is determined by some constants where $t \to k_T t$, $u \to k_U u$, and $v \to k_V v$, for $t \in T$, $u \in U$, $v \in V$.

Back to Cyclic Orders

As a result of the previous slide, we have the following fact.

6×6 matrix form

Every matrix *must* have the following form to assure neutrality.

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More facts are now fairly easy to come by.

Theorem

For a 'Borda-like' system, the set of profiles which do not impact the election at all (kernel) is $T \oplus V$.
For a 'vote for one and its reversal' system, the kernel is U.

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6 × 6 matrix form

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$$\begin{pmatrix} a & b & c & c & c & c \\ b & a & c & c & c & c \\ c & c & a & b & c & c \\ c & c & b & a & c & c \\ c & c & c & c & a & b \\ c & c & c & c & b & a \end{pmatrix}$$

More facts are now fairly easy to come by.

Theorem

For a 'Borda-like' system, the set of profiles which do not impact the election at all (kernel) is $T \oplus V$.
For a 'vote for one and its reversal' system, the kernel is U.

There are nice results for $n = 5$ too, but we will skip them for now.

Outline

Where's the experimenting?

Let's take stock.

- We've connected voting and algebra.
- We've introduced cyclic orders, and voting on them.
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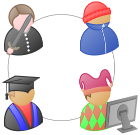
So let's see where this enters in the story.

More interesting ballots

Consider once again the poker-type example.

Could we have a ballot that gives a little more agency than just picking your favorite order?



Granted that you are betting on *one* player[1], it's reasonable to say you would want to put a weak player on their left and a stronger player on their right.

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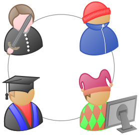
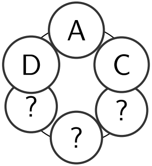


Granted that you are betting on *one* player[1], it's reasonable to say you would want to put a weak player on their left and a stronger player on their right.

In this case, we can suggest ballots that look like $A\{D, C\}$, where we interpret this as desiring A to have D its right, and C to its left. (We assume play goes clockwise.)



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More interesting ballots

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In this case, we can suggest ballots that look like $A\{D, C\}$, where we interpret this as desiring A to have D its right, and C to its left. We call this a ROLO ballot.

- With four agents, ROLO has *twenty-four* total possible ballots: $\{A\{D, C\}, B\{C, D\}, \ldots, A\{B, C\}\}$
- The outcome space should still be the set of six cyclic orders, and each ballot clearly indicates a preferred outcome (e.g. $A\{D, C\}$ would like to see $ACBD$).
- However, we now know more details about *why* each voter prefers a particular order!

More interesting ballots

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The ROLO(2,1) procedure takes a ROLO ballot $A\{D,C\}$, and then gives two points to $ACBD$, but one point each to $ACDB$ and $ABCD$.

Here is the full matrix for ROLO(2,1).

$$\begin{pmatrix} 2&2&2&2&0&0&0&0&1&0&0&1&1&0&1&0&1&0&1&0&1&0&0&1\\ 0&0&0&0&2&2&2&2&0&1&1&0&0&1&0&1&0&1&0&1&0&1&1&0\\ 1&0&1&0&1&0&0&1&2&2&2&2&0&0&0&0&0&1&1&0&1&0&1&0\\ 0&1&0&1&0&1&1&0&0&0&0&0&2&2&2&2&1&0&0&1&0&1&0&1\\ 0&1&1&0&1&0&1&0&1&0&1&0&0&1&1&0&2&2&2&2&0&0&0&0\\ 1&0&0&1&0&1&0&1&0&1&0&1&1&0&0&1&0&0&0&0&2&2&2&2 \end{pmatrix}$$

It's a handful, but still usable, and very similar theorems about decomposition of the profile and outcome spaces can be had. But ...

Finding Ties

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We *can* investigate what *nonnegative integer* profiles lead to complete ties between all six possible cyclic orders. The highly symmetric nature of the matrix makes it easy to pick out examples.

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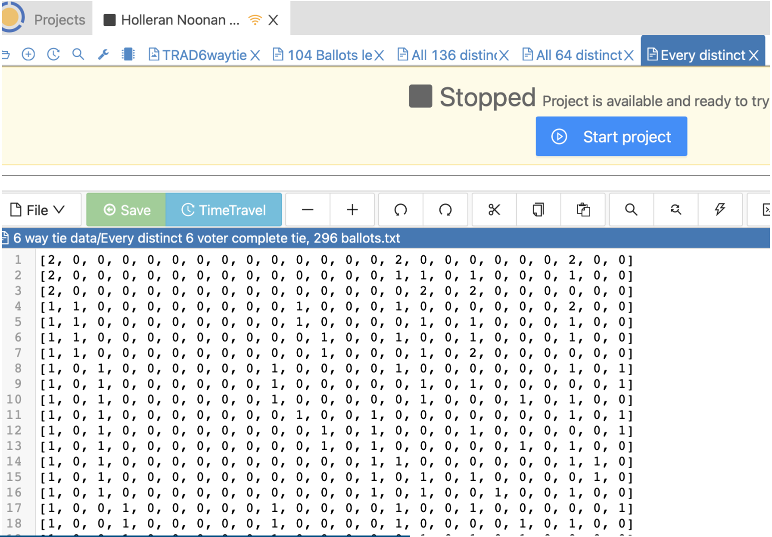
We *can* investigate what *nonnegative integer* profiles lead to complete ties between all six possible cyclic orders. The highly symmetric nature of the matrix makes it easy to pick out examples. But representation theory *cannot* answer directly what all of them must be, since it only deals with vector spaces. Here, experimentation was crucial – particularly for students who didn’t yet know any group theory to be able to essentially reconstruct the action of the symmetric group, without knowing that was what they were doing!

Finding Ties

Raw data:



Finding Ties

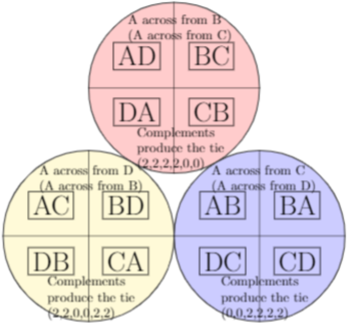
Organizing data by permutations:

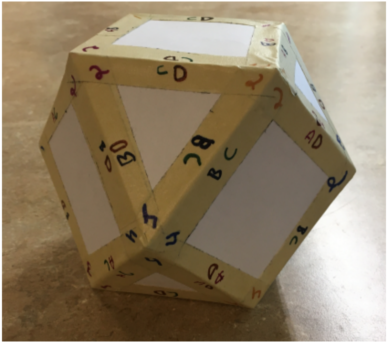
Across	Right	Across	Right	Across	right
AxB	C-B	AxC	C-D	AxD	C-A
AxB	A-C	AxC	B-C	AxD	D-C
AxB	C-B	AxC	A-B	AxD	D-B
AxB	B-D	AxC	B-C	AxD	B-A
AxB	D-A	AxC	B-A	AxD	C-A
AxB	A-C	AxC	A-D	AxD	A-B
AxB	D-A	AxC	D-C	AxD	D-B
AxB	B-D	AxC	A-D	AxD	C-D
AxB	A-D	AxC	C-D	AxD	B-D
AxB	D-B	AxC	D-A	AxD	D-C
AxB	A-D	AxC	A-B	AxD	A-C
AxB	C-A	AxC	D-A	AxD	B-A
AxB	B-C	AxC	B-A	AxD	B-D
AxB	D-B	AxC	C-B	AxD	A-B
AxB	B-C	AxC	D-C	AxD	A-C
AxB	C-A	AxC	C-B	AxD	C-D

Across	Right	Across	Right	Across	right
AxB	D-B	AxC	D-A	AxD	D-C
AxB	A-D	AxC	C-D	AxD	B-D
AxB	A-C	AxC	B-C	AxD	D-C
AxB	C-B	AxC	C-D	AxD	C-A
AxB	D-B	AxC	C-B	AxD	A-B
AxB	B-C	AxC	B-A	AxD	B-D
AxB	A-C	AxC	A-D	AxD	A-B
AxB	D-A	AxC	B-A	AxD	C-A
AxB	C-A	AxC	D-A	AxD	B-A
AxB	A-D	AxC	A-B	AxD	A-C
AxB	B-D	AxC	B-C	AxD	B-A
AxB	C-B	AxC	A-B	AxD	D-B
AxB	C-A	AxC	C-B	AxD	C-D
AxB	B-C	AxC	D-C	AxD	A-C
AxB	B-D	AxC	A-D	AxD	C-D
AxB	D-A	AxC	D-C	AxD	D-B

Finding Ties

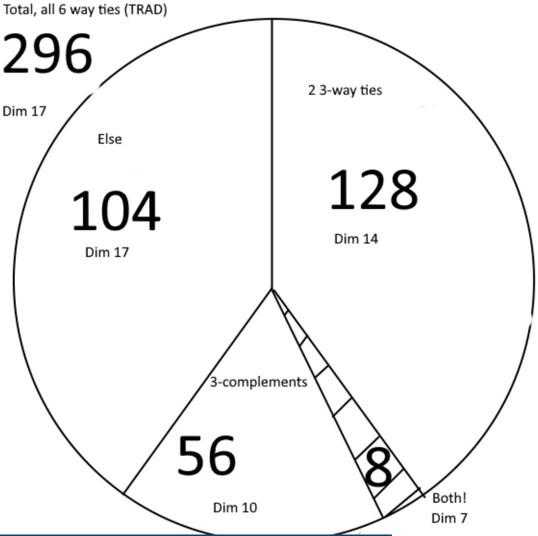
Graphically organizing data:





Finding Ties

Analyzing data:



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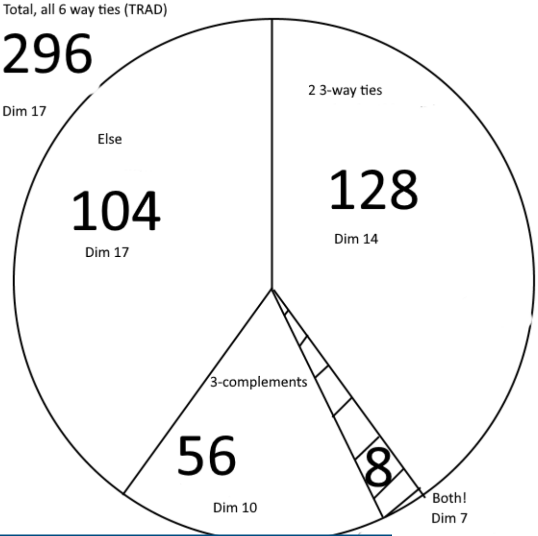

There are still dimensions missing! There is still a lot of room for research into what form all complete ties must take, though my students have moved on to other pursuits.

Acknowledgments

Thanks are due to:

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- The organizers of this online seminar
- Gordon College Provost's Summer Undergraduate Research Fellowship, and especially my students Abraham Holleran, Micah Martin, and Josephine Noonan
- Mike Orrison, Niles Johnson, and Bill Zwicker for useful comments and support
- You for listening!

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Number Five is Alive

Cyclic Orders for $n = 5$

For $\mathcal{A} = \{A, B, C, D, E\}$ we have the following 24 options:

$$\{ABCDEA, AEDCBA, ABCEDA, ADECBA, ABDCEA, AECDBA, ABDECA, ACEDBA, ABECDA, ADCEBA, ABEDCA, ACDEBA, ACBDEA, AEDBCA, ACDBEA, AEBDCA, ACEBDA, ADBECA, ADBCEA, AECBDA, AEBCDA, ADCBEA, ACBEDA, ADEBCA\}$$

For $n = 5$, characters help decompose the space as

$$T_5 \oplus S \oplus H_1 \oplus H_2 \oplus J^2$$

where each of these subspaces is irreducible (of dimensions) 1, 1, 5, 5, and 6 (squared), respectively. These spaces *do* have meaning in terms of the voting profiles.

Number Five is Alive

Every procedure comes from this 24 × 24 matrix

$$\begin{pmatrix}
a & b & c & d & c & d & e & f & e & f & d & c & c & d & e & f & g & h & e & f & c & d & f & e \\
b & a & d & c & d & c & f & e & f & e & c & d & d & c & f & e & h & g & f & e & d & c & e & f \\
c & d & a & b & e & f & d & c & c & d & e & f & f & e & g & h & e & f & c & d & e & f & c & d \\
d & c & b & a & f & e & c & d & d & c & f & e & e & f & h & g & f & e & d & c & f & e & d & c \\
c & d & e & f & a & b & c & d & d & c & e & f & e & f & d & c & e & f & c & d & f & e & h & g \\
d & c & f & e & b & a & d & c & c & d & f & e & f & e & c & d & f & e & d & c & e & f & g & h \\
e & f & d & c & c & d & a & b & e & f & c & d & c & d & f & e & d & c & f & e & h & g & f & e \\
f & e & c & d & d & c & b & a & f & e & d & c & d & c & e & f & c & d & e & f & g & h & e & f \\
e & f & c & d & d & c & e & f & a & b & c & d & h & g & e & f & d & c & f & e & d & c & e & f \\
f & e & d & c & c & d & f & e & b & a & d & c & g & h & f & e & c & d & e & f & d & c & f & e \\
d & c & e & f & e & f & c & d & c & d & a & b & f & e & d & c & f & e & h & g & f & e & c & d \\
c & d & f & e & f & e & d & c & d & c & b & a & e & f & c & d & e & f & g & h & e & f & d & c \\
c & d & f & e & e & f & c & d & g & h & f & e & a & b & c & d & e & f & d & c & f & e & c & d \\
d & c & e & f & f & e & d & c & h & g & e & f & b & a & d & c & f & e & c & d & e & f & d & c \\
e & f & h & g & d & c & f & e & e & f & d & c & c & d & a & b & d & c & e & f & d & c & e & f \\
f & e & g & h & c & d & e & f & f & e & c & d & d & c & b & a & c & d & f & e & c & d & f & e \\
h & g & e & f & e & f & d & c & d & c & f & e & e & f & d & c & a & b & d & c & e & f & c & d \\
g & h & f & e & f & e & c & d & c & d & e & f & f & e & c & d & b & a & c & d & f & e & d & c \\
e & f & c & d & c & d & f & e & f & e & g & h & d & c & e & f & d & c & a & b & d & c & f & e \\
f & e & d & c & d & c & e & f & e & f & h & g & c & d & f & e & c & d & b & a & c & d & e & f \\
c & d & e & f & f & e & g & h & c & d & f & e & f & e & d & c & e & f & d & c & a & b & d & c \\
d & c & f & e & e & f & h & g & d & c & e & f & e & f & c & d & f & e & c & d & b & a & c & d \\
f & e & c & d & g & h & f & e & e & f & c & d & c & d & e & f & c & d & f & e & d & c & a & b
\end{pmatrix}$$