A heuristic link between divisor counts and prime densities in sequences

Benoît Cloitre

Rutgers Experimental Mathematics Seminar

Thu., Sept. 18, 2025

The arithmetic tableau

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
1		2		3		4		5		6		7		8		9		10		11		12		13		14		15		16
1			2			3			4			5			6			7			8			9			10			11
1				2				3				4				5				6				7				8		
1					2					3					4					5					6					7
1						2						3						4						5						6
1							2							3							4							5		
1								2								3								4						
1									2									3									4			
1										2										3										4
1											2											3								
1												2												3						
1													2													3				
1														2														3		
1															2															3
1																2														
1																	2													

In the sequel, we remove the 1's since they are not proper divisors.

Example: $m = 12$ (composite)

The SW diagonal hits the lattice points corresponding to proper divisors: (line 2=6), (line 3=4), (line 4=3), (line 6=2).

2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
	2		3		4		5		6		7		8		9		10		11		12		13		14		15		
		2			3			4			5			6			7			8			9			10			
			2				3				4				5				6				7						
				2					3					4					5					6					
					2						3						4						5						
						2							3							4									
							2								3														
								2									3												
									2										3										
										2																			
											2																		
												2																	
													2																

Example: $m = 17$ (prime)

The SW diagonal starting from 17 hits no lattice points.

This is the visual signature of a prime in the tableau.

2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
	2		3		4		5		6		7		8		9		10		11		12		13		14		15		
		2			3			4			5			6			7			8			9			10			
			2				3				4				5				6				7						
				2					3					4					5					6					
					2						3						4						5						
						2							3							4									
							2								3														
								2									3												
									2										3										
										2																			
											2																		
												2																	
													2																

A heuristic idea: The Divisor Mass Ratio

Consider the total divisor mass up to N:

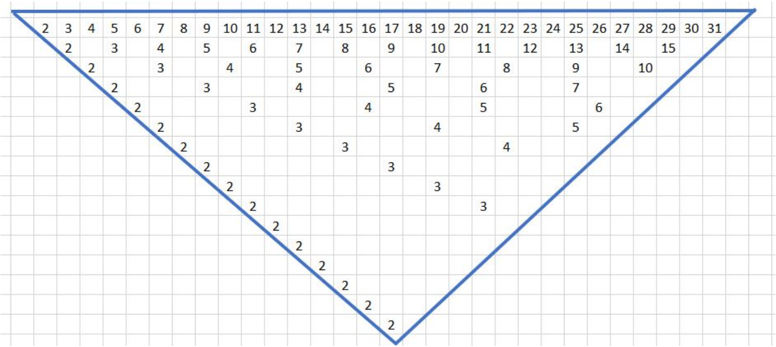
$$s(N) = \sum_{m \leq N} \tau(m).$$

We define the *Divisor Mass Ratio* (as $N \to \infty$):

$$R(N) = \frac{N}{s(N)}.$$

- **Heuristic interpretation:** a gauge of how sparse proper divisors are in the arithmetic tableau; it mimics the probability of avoiding a proper-divisor hit when randomly sampling a filled cell in the triangular grid below (here $N = 31$).

The heuristic interpretation of R(N)



The Central Question

Because "avoiding any proper divisor from $\{1, \ldots, N\}$" intuitively feels like "hitting a prime," we compare the *Divisor Mass Ratio* to the *Prime Density* on $\{1, \ldots, N\}$:

$$\Pi(N) = \frac{\pi(N)}{N}.$$

Question: Does the proxy predict the prime density?

$$R(N) = \frac{N}{\sum_{m \leq N} \tau(m)} \sim \frac{\pi(N)}{N} = \Pi(N) \quad ?$$

Baseline check: The Prime Number Theorem

Classical asymptotics:

- **Divisor Side (Dirichlet's divisor sum):** $\sum_{m \leq N} \tau(m) \sim N \log N$.

$$R(N) \sim \frac{N}{N \log N} = \frac{1}{\log N}.$$

- **Prime Side (PNT):**

$$\Pi(N) \sim \frac{1}{\log N}.$$

Hence, $R(N) \sim \Pi(N)$. Is this just a coincidence, or a structural link?

Test Case: Arithmetic Progressions (APs)

Let $u(k) = ak + b$ with $\gcd(a, b) = 1$. Let $s_u(n) = \sum_{k=1}^{n} \tau(u(k))$ be the divisor mass for the first n terms and $R_u(n) = n/s_u(n)$.

- **Divisor Side:** Rigorous analysis shows

$$s_u(n) \sim \frac{\varphi(a)}{a} n \log n.$$

- Hence

$$R_u(n) \sim \frac{a}{\varphi(a)} \cdot \frac{1}{\log n}.$$

- **Prime Side (PNT in AP):**

$$\Pi_u(n) = \frac{\pi_u(n)}{n} \sim \frac{a}{\varphi(a)} \cdot \frac{1}{\log n}.$$

Therefore we have again $R_u(n) \sim \Pi_u(n)$.

Test Case: Residue Slices

Fix M and a subset of invertible classes $\mathcal{C} \subseteq (\mathbb{Z}/M\mathbb{Z})^{\times}$. Let U_n be the first n integers lying in $\mathcal{C}$.

- **Divisor side:**

$$s_u(n) \sim \frac{\#\mathcal{C}}{M} n \log n \quad \Rightarrow \quad R_u(n) = \frac{n}{s_u(n)} \sim \frac{M}{\#\mathcal{C}} \cdot \frac{1}{\log n}.$$

- **Prime side:**

$$\Pi_u(n) = \frac{\pi_u(n)}{n} \sim \frac{M}{\varphi(M)} \cdot \frac{1}{\log n}.$$

Conclusion. $\Pi_u(n) \sim \underbrace{\frac{\#\mathcal{C}}{\varphi(M)}}_{=:w \text{ (constant)}} R_u(n)$. In particular, if $\mathcal{C} = (\mathbb{Z}/M\mathbb{Z})^{\times}$ then the proxy matches exactly.

Other balanced families

Examples where the first-order match persists i.e. $R_u(n) \sim \Pi_u(n)$.

- $u(k) = \lfloor \alpha k \rfloor$, $\alpha > 1$ irrational (Beatty sequence).
- $u(k) = k + \lfloor \sqrt{k} \rfloor$.
- Mild inhomogeneities with bounded gaps and stable residue statistics.

Admissibility and growth

We restrict the scope to sequences $u(k)$ satisfying:

- **Admissibility:** No fixed prime divides all large values.
- **Moderate growth:** Linear or polynomial growth.
- **Regularity:** Stable distribution in invertible classes mod some M.

Conjecture 1: Balanced Continuity

Conjecture A (Balanced Continuity): For admissible, regular

sequences of moderate growth with stable local properties, the prime density equals the divisor mass ratio asymptotically.

$$\Pi_u(n) \sim R_u(n).$$

$$\frac{\pi_u(n)}{n} \sim \frac{n}{s_u(n)}.$$

Stable Continuity: Shifted primes $u(k) = p_k + 2$

We examine the density of primes in the sequence of shifted primes (related to Twin Primes).

- **Divisor Side (Titchmarsh Divisor Problem):**

$$s_u(n) = \sum_{k \le n} \tau(p_k + 2) \sim C_3\, n \log n.$$

  Proxy: $R_u(n) \sim 1/(C_3 \log n)$.

- **Prime Side (Hardy-Littlewood):**

$$\Pi_u(n) \sim \frac{K_3}{\log n}, \quad K_3 = 2\, C_{\text{twin}}.$$

The scale is correct, but the constants differ. $\Pi_u(n) \sim L \cdot R_u(n)$. The multiplier is $L = K_3/C_3$ (Stable, Biased case).

Stable Continuity: Quadratic values $u(k) = k^2 + 1$

Related to Landau's 4th problem.

- **Divisor Side (Hooley, McKee):**

$$s_u(n) = \sum_{k \le n} \tau(k^2 + 1) \sim C_4\, n \log n.$$

  $C_4 > 0$ is explicit (Euler product).

- **Prime Side (Bateman–Horn):**

$$\Pi_u(n) \sim \frac{K_4}{\log n}.$$

Again a stable, bounded multiplier $L = K_4/C_4$.

Conjecture 2: Stable Continuity

Conjecture B (Stable Continuity): For admissible sequences

corresponding to stable configurations (e.g., fixed polynomials), there exists a constant $L > 0$ such that

$$\Pi_u(n) \sim L \cdot R_u(n).$$

If $L = 1$, it is balanced (Conjecture A). If $L \neq 1$, it is stable (biased).

Oscillatory Continuity: Goldbach Slices (Divisor Side)

We examine the family $U_N = \{N - p : p \leq N, p \text{ prime}\}$. Let $N = 2n$. The length is $\nu = \pi(N)$.
Divisor Side (Goldbach Divisor Problem): The divisor mass $S(N) = \sum_{p \leq N} \tau(N - p)$ is known rigorously:

$$S(N) \sim C_5(N)\, N.$$

$C_5(N)$ oscillates based on the factorization of N:

$$C_5(N) = C_0 \prod_{p|N} \frac{(p-1)^2}{p^2 - p + 1}, \quad C_0 = \frac{\zeta(2)\zeta(3)}{\zeta(6)}.$$

Proxy: $R(N) = \frac{\pi(N)}{S(N)} \sim \frac{1}{C_5(N)\,\log N}$.

Goldbach Slices: Prime Side and the factor $w(N)$

Prime Side (Hardy-Littlewood): The prime count $G(N)$ is conjectured:

$$G(N) \sim \mathfrak{S}(N) \, \frac{N}{(\log N)^2}.$$

Prime Density: $\Pi(N) = \frac{G(N)}{\pi(N)} \sim \frac{\mathfrak{S}(N)}{\log N}$.

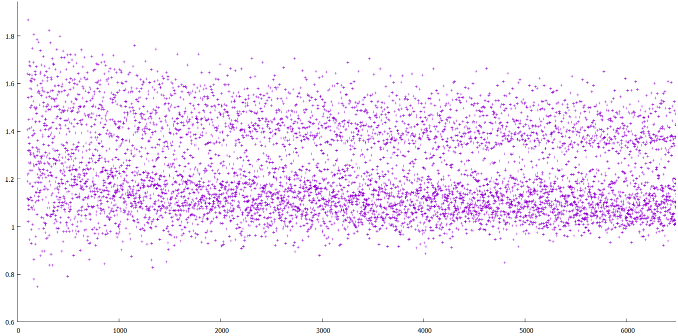
The Continuity Factor $w(N)$:

$$\Pi(N) \sim w(N) \cdot R(N).$$

$$w(N) \sim \mathfrak{S}(N) \cdot C_5(N).$$

$w(N)$ oscillates but is rigorously bounded.

Goldbach: plot of $w(N)$



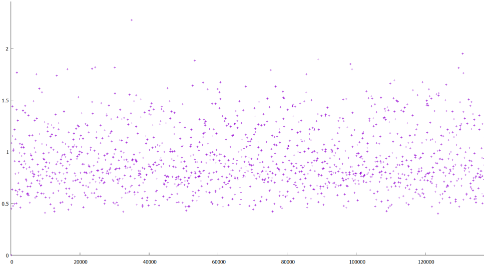
Oscillatory Continuity: Inhomogeneous Squares

Family $U_n = \{n + k^2 : 1 \le k \le n\}$.

- Divisor mass involves a parameter-dependent local factor $C_6(n)$.
- Prime side (Bateman-Horn) involves a parameter-dependent singular series $K_6(n)$.

The ratio $w(n) = K_6(n)/C_6(n)$ is bounded but oscillatory.

Plot of $w(n)$ for $n + k^2$.



Conjecture 3: Oscillatory Continuity

Conjecture C (Oscillatory/General Continuity): For parametrised admissible families U_n of moderate growth,

$$\Pi_U(n) \sim w(n) \cdot R_U(n),$$

with $w(n)$ strictly bounded away from 0 and ∞, but not necessarily convergent.

$$0 < \liminf w(n) \leq \limsup w(n) < \infty.$$

A Domination Principle ($k = 2$)

Even without knowing $w(n)$, we propose a weaker principle.
Conjecture D (Domination, $k = 2$): For large n:

$$\Pi_U(n) \geq (R_U(n))^2 .$$

$$\frac{\pi_U(n)}{\nu} \geq \left(\frac{\nu}{s_U(n)}\right)^2 .$$

Consequence: If the divisor mass grows typically, $s_U(n) \sim C\,\nu \log \nu$, then $\pi_U(n) \gg \nu/(\log \nu)^2 \to \infty$. This forces infinitely many primes.

Domination Principle (general k)

Conjecture D (General Domination): More generally, for some integer $k \geq 2$ and large n:

$$\Pi_U(n) \geq (R_U(n))^k \,.$$

This parallels the structure of m-tuple conjectures. It provides a path to proving infinitude without exact constants.

What is proved vs conjectured

- **Divisor side (Provable):** Asymptotics for the divisor mass $s(n)$ (including constants) can often be established rigorously (Dirichlet, APs, Titchmarsh, Goldbach Divisor Problem).
- **Prime side (Often Conjectural):** PNT/AP are theorems, but HL, BH, and Goldbach are conjectures.
- **The Link:** The comparison (Conjectures A, B, C, D) is heuristic but demonstrably consistent with known results and explicit constants.

Takeaways

- A simple Divisor Mass Ratio (R_U) predicts prime frequencies (Π_U) at first order.
- The geometry of the arithmetic tableau explains the $1/\log$ scale.
- Three regimes: Balanced (constants match), Stable (bounded multiplier L), Oscillatory (bounded $w(n)$).
- The proxy is robust because the divisor mass is often rigorously computable.

References (minimal)

Classical Texts: Davenport; Montgomery–Vaughan; Iwaniec–Kowalski; Tenenbaum. **Divisor Problems:** Titchmarsh (1930); Linnik (Dispersion method); Bombieri–Vinogradov; Hooley (Quadratic polynomials). **Prime Conjectures:** Hardy–Littlewood (1923, Partitio Numerorum); Bateman–Horn (1962).

Thanks

Thank you for your attention.

Happy to take questions.