Multislice Spectral Gap and Gap Eigenfunctions

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A physical model

Given an integer $N \geq 2$ and a set $\{e_0, \ldots, e_{r-1}\} \subset \mathbb{R}$, $r \geq 2$, consider a graph whose vertices are $\{e_0, \ldots, e_{r-1}\}^N$; i.e., the N-tuples $x = (x_1, \ldots, x_N)$ where $x_j \in \{e_0, \ldots, e_{r-1}\}$ for each j. In the application that motivates this investigation, $\{e_0, \ldots, e_{r-1}\}$ is the set of "energy levels" an individual particle may have in a system of N particles. The total energy of the system in the state x is then

$$E(x) := \sum_{\ell=1}^{N} x_\ell \ .$$

Consider a random walk on $\{e_0, \ldots, e_{r-1}\}^N$ where at each step, a pair (i, j), $1 \leq i < j \leq N$ is selected, and then the particles i and j "collide", and x_i and x_j are updated to new values x_i' and x_j' in $\{e_0, \ldots, e_{r-1}\}$ such that

$$x_i + x_j = x_i' + x_j' \ ,$$

conserving the total energy.

We now assume that $\{e_0, \ldots, e_{r-1}\}^N$ is linearly independent over the rationals. This has two consequences:

First, for any $0 \leq m_1, m_2, m_3, m_4 \leq r - 1$,

$$e_{m_1} + e_{m_2} = e_{m_3} + e_{m_4} \iff \{m_1, m_2\} = \{m_3, m_4\} \ .$$

Second, two states x and y have the same energy if and only if, for each $0 \leq m \leq r-1$, the numbers of entries of x and y that are e_m are the same. Call this number k_m. Thus the level sets of the energy function $E(x)$ are indexed by the set of vectors $\mathbf{k} = (k_0, \ldots, k_{r-1})$ such that each k_m is a non-negative integer and

$$\sum_{m=0}^{r-1} k_m = N \ .$$

We define $\mathcal{V}_{N,\mathbf{k}}$ to be the set of all such $x \in \{e_0, \ldots, e_{r-1}\}^N$.

$\mathcal{V}_{N,\mathbf{k}}$ is called a *multislice*. It is a mult-level analog of a slice of the Boolean cube $\{0,1\}^N$.

The graph $\mathcal{G}_{N,\mathbf{k}}$

The pair interaction between particles induces a notion of adjacency in $\{e_0, \dots, e_{r-1}\}^N$ making it a graph. Under the non-degeneracy condition, the only new vertices to which jumps from x are possible are those of the form $\pi_{i,j}x$ where

$$(\pi_{i,j}x)_\ell = \begin{cases} x_j & \ell = i \\ x_i & \ell = j \\ x_\ell & \ell \neq i,j \end{cases} .$$

Two *distinct* vertices x and y are adjacent if for some $i < j$, $y = \pi_{i,j}x$, in which case evidently $x = \pi_{i,j}y$ as well.

Thus, in the graph we have just defined, whose vertex set is $\{e_0, \dots, e_{r-1}\}^N$ and in which adjacency is defined through pair transpositions as described above, the connected components are precisely the graphs $\mathcal{G}_{N,\mathbf{k}}$ whose vertex set is $\mathcal{V}_{N,\mathbf{k}}$.

Evidently, the cardinality of $\mathcal{V}_{N,\mathbf{k}}$ is

$$\frac{N!}{k_0! \cdots k_{r-1}!} \ .$$

Let $\mathcal{E}_{N,\mathbf{k}}$ denote the edge set; i.e., the set of all pairs $\{x, y\}$ of distinct vertices that are adjacent. Given $x \in \mathcal{V}_{N,\mathbf{k}}$ and $0 \le m < n \le r-1$, there are $k_m k_n$ pair transpositions that swap energy levels e_m and e_n. Summing over $m < n$, we obtain the number of adjacent vertices; i.e., the degree $\delta_{N,\mathbf{k}}$:

$$\delta_{N,\mathbf{k}} = \sum_{m<n} k_m k_n \ .$$

Evidently all of these graphs are regular.

The graph Laplacian $L_{\mathcal{G}}$

Given a finite undirected graph $\mathcal{G}$ with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$, the *graph Laplacian* $L_{\mathcal{G}}$ is the operator on functions f on $\mathcal{V}$ given by

$$L_{\mathcal{G}}f(x) = \sum_{y\in\mathcal{V}\ :\ \{x,y\}\in\mathcal{E}} (f(x) - f(y))\ .$$

Note that

$$\sum_{x\in\mathcal{V}} f(x)L_{\mathcal{G}}f(x) = \sum_{y\in\mathcal{V}\ :\ \{x,y\}\in\mathcal{E}} \frac{1}{2}(f(x) - f(y))^2\ .$$

Therefore, if $\mu_{\mathcal{V}}$ denotes the uniform probability measure on $\mathcal{V}$, $L_{\mathcal{G}}$ is a positive semi-definite operator on $L^2(\mu_{\mathcal{V}})$, and that $L_{\mathcal{G}}f = 0$ if and only if f is constant on each connected component of $\mathcal{G}$. Hence, on a connected graph $\mathcal{G}$, 0 is an eigenvalue of multiplicity one, and the eigenspace is spanned by the constant vector.

Note that with δ denoting the degree,

$$L_{\mathcal{G}} = \delta I - A_{\mathcal{G}} \ ,$$

where $A_{\mathcal{G}}$ is the *adjacency* matrix of the graph.

Definition

Let $\mathcal{G}$ be a finite connected graph with vertex set $\mathcal{V}$. The *spectral gap* of $\mathcal{G}$, $\Gamma_{\mathcal{G}}$, is the least non-zero eigenvalue of $L_{\mathcal{G}}$.

By the Rayleigh-Ritz variational principle,

$$\Gamma_{\mathcal{G}} = \inf\left\{ \int_{\mathcal{V}} f(x) L_{\mathcal{G}} f(x) \mathrm{d}\mu_{\mathcal{V}} \ : \ \int_{\mathcal{V}} f(x)\mathrm{d}\mu_{\mathcal{V}} = 0 \ , \ \int_{\mathcal{V}} |f(x)|^2 \mathrm{d}\mu_{\mathcal{V}} = 1 \ \right\} \ .$$

Some special cases

(1) if $k_m = N$ for some $0 \leq m \leq r-1$, then $\mathcal{V}_{N,\mathbf{k}}$ is a singleton. The edge set $\mathcal{E}_{N,\mathbf{k}}$ is empty, $L_{\mathcal{G}_{N,\mathbf{k}}} = 0$, and has no gap.

(2) If $k_{m_0} = N-1$ for some m_0, then $k_{m_1} = 1$ for one value of $m_1 \neq m_0$, and $k_n = 0$ for all $n \neq m_0, m_1$. The graph is complete N vertices, and therefore the spectral gap is N.

(3) If $r = N$ and $\mathbf{k} = (1, \dots, 1)$, then $\mathcal{V}_{N,(1,\dots,1)}$ has $N!$ vertices and may be identified with S_N, the symmetric group on N letters. The spectrum of the corresponding graph Laplacian $L_{\mathcal{V}_{N,(1,\dots,1)}}$ has been studied using methods from group representation theory by Diaconis and Shahshahani. One of their results is that for all N, the spectral gap is N.

(4) For $r = 2$, we might as well take $\{e_0, e_1\} = \{0, 1\}$, and $\mathcal{G} = \{0, 1\}^N$, the Boolean N-cube, with adjacency defined as above. The connected components $\mathcal{G}_{N(k_0,k_1)}$ are known as *Johnson Graphs*, and the full spectrum of the Laplacian $L_{\mathcal{G}_{N,(k_0,k_1)}}$ is known, along with all eigenvalues. In particular, it is known that the spectral gap is always N independent of $\mathbf{k} = (k_0, k_1)$ assuming that both k_0 and k_1 are non-zero.

This appears to be the last well-studied case. For the remaining multislices, there is some spectral informations but little information on eigenvectors.

An example

Consider, as an expeimental warm-up, $\mathcal{G}_{4,(2,1,1)}$. This is the only "new" graph that shows up among the conected components of $\{e_0, e_1, e_2\}^4$.

There are $\binom{4}{2} = 6$ ways to place the 2 e_0 entries in x, and then two ways to place the remaining entries, so there are 12 vertices. The valency is $2 + 2 + 1 = 5$.

The 12 by 12 adjacency matrix can be worked out and then diagonalized in Maple.

Eigenvectors and eigenvalues

$$
> \left[\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -3 \\ -3 \\ -3 \\ -1 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix}
1 & 1 & 0 & -1 & -1 & 1 & 0 & 1 & -1 & -1 & -1 & 1 \\
0 & 0 & -1 & -1 & 0 & -1 & 1 & 1 & -1 & 0 & 1 & 1 \\
-1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & -1 & -1 & -1 & 1 \\
0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 \\
-1 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & -1 & -1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & -1 & -1 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1
\end{bmatrix} \right]
$$

Definition

Let $\mathcal{K}_{N,\mathbf{k}}$ denote the set of real-valued functions g on $\{e_0, \dots, e_{r-1}\}$ such that

$$\sum_{m=0}^{r-1} k_m g(e_m) = 0 \ .$$

Note that for $\mathcal{K}_{4,(2,1,1)}$ is two dimensional and is spanned by the functions corresponding to the vectors

$$(1,-1,-1) \qquad \text{and} \qquad (0,1,-1) \ .$$

Main result

Theorem

For all $N \geq 2$, all $r \geq 2$, and all $\mathbf{k} = (k_0, \ldots, k_{r-1}) \in \mathbb{Z}^r_{\geq 0}$ with

$$\sum_{m=0}^{r-1} k_m = N \qquad \text{and} \qquad \max\{k_0, \ldots, k_{r-1}\} < N \ ,$$

so that $\mathcal{G}_{N,\mathbf{k}}$ is not trivial, the spectral gap $\Gamma_{N,\mathbf{k}}$ of $L_{\mathcal{G}_{N,\mathbf{k}}}$ is given by

$$\Gamma_{N,\mathbf{k}} = N \ .$$

The eigenspace of $L_{\mathcal{G}_{N,\mathbf{k}}}$ with eigenvalue N has the dimension $(N-1)(r-1)$ and for any basis $\{g_1, \ldots g_{r-1}\}$ of $\mathcal{K}_{N,\mathbf{k}}$, the set

$$\{g_m(x_\ell) \ ; 1 \leq m \leq r-1, \quad 1 \leq \ell \leq N-1 \ \}$$

is a basis for the gap eigenspace of $L_{N,\mathbf{k}}$.

Lemma

Let $g \in \mathcal{K}_{N,\mathbf{k}}$, $1 \le \ell \le N$. Define a function f on $\mathcal{V}_{N,\mathbf{k}}$ by $f(x) = g(x_\ell)$. Then

$$L_{\mathcal{G}_{N,\mathbf{k}}} f(x) = N f(x) \ .$$

Proof: Note that $f(x) - f(\pi_{i,j}x) = 0$ unless $i = \ell$ or $j = \ell$. For any $0 \le m \le r-1$, consider $x \in \mathcal{V}_{N,\mathbf{k}}$ such that $x_\ell = e_m$. For each $n \ne m$, there are k_n pair permutations such that when applied to x yield the value e_n in the ℓth place. Therefore, since $\sum_{n=0}^{r-1} k_n g(e_n) = 0$

$$\begin{aligned} L_{\mathcal{G}_{N,\mathbf{k}}} f(x) &= \sum_{n \ne m} k_n (g(e_m) - g(e_n)) \\ &= (N - k_m) g(e_m) - \sum_{n \ne m} k_n g(e_n) \\ &= (N - k_m) g(e_m) + k_m g(e_m) = N g(e_m) \ . \end{aligned}$$

Since m is arbitrary, the lemma is proved.

Fix any $g \in \mathcal{K}_{M,\mathbf{k}}$. Then the N functions $\{g(x_1), \dots, g(x_N)\}$ are not linearly independent since for any $x \in \mathcal{V}_{N,\mathbf{k}}$,

$$\sum_{\ell=1}^{N} g(x_\ell) = \sum_{n=0}^{r-1} k_n g(e_n) = 0 \ .$$

For any $\{g_1, \dots, g_N\} \subset \mathcal{K}_{M,\mathbf{k}}$ define $f(x) = \sum_{\ell=0}^{N} g_\ell(x_\ell)$. By Lemma 4, $L_{\mathcal{G}_{N,\mathbf{k}}} f = Nf$. However, we can express f in a simpler way: Since $\sum_{\ell=1} g_N(x_\ell) = 0$,

$$f(x) = f(x) - \left(\sum_{\ell=1} g_N(x_\ell)\right) = \sum_{\ell=1}^{N-1} h_\ell(x_\ell)$$

where for $1 \leq \ell \leq N-1$, $h_\ell = g_\ell - g_N$. It is essentially for this reason that the index ℓ only ranges over $\{1, \dots, N-1\}$ in the theorem.

The induction for the lower bound on the gap

Define $\mu_{N,\mathbf{k}} := \left(\frac{N!}{k_0! \cdots k_{r-1}!}\right)^{-1}$. We also write $\mu_{N,\mathbf{k}}$ to denote the uniform probability measure on $\mathcal{V}_{N,\mathbf{k}}$.
The Dirichlet form for $L_{\mathcal{G}_{N,\mathbf{k}}}$ on $L^2(\mu_{N,\mathbf{k}})$ is

$$\frac{1}{2} \sum_{x \in \mathcal{V}_{N,\mathbf{k}}} \sum_{i<j} (f(x) - f(\pi_{i,j}x))^2 \mu_{N,\mathbf{k}} \ .$$

It greatly simplifies the induction we shall carry out if we introduce more averaging, and consider

$$\frac{1}{2} \binom{N}{2}^{-1} \sum_{x \in \mathcal{V}_{N,\mathbf{k}}} \sum_{i<j} (f(x) - f(\pi_{i,j}x))^2 \mu_{N,\mathbf{k}} \ .$$

The key to our induction is the identity

$$\binom{N}{2}^{-1} \sum_{i<j} (f(\pi_{i,j}x) - f(x))^2 =$$

$$\frac{1}{N} \sum_{\ell=1}^{N} \left(\binom{N-1}{2}^{-1} \sum_{i<j, i,j \neq \ell} (f(\pi_{i,j}x) - f(x))^2 \right) ,$$

On the right, we have an average of N terms, each of which leave on one coordinate, x_ℓ, unchanged – transpositions $\pi_{i,j}$ in which either $i = \ell$ or $j = \ell$ are not included. If one thinks in terms of processes defined by the Dirichlet forms, this identity will relate the dynamics for N particles to the dynamics for $N - 1$ particles.

We make one more adjustment: The latest Dirichlet form is associated to a continuous time Markov jump process on $\mathcal{V}_{N,\mathbf{k}}$ of the following description: A Poisson clock is running with expected times between "rings" equal to 2. When a "ring" occurs, a pair (i,j), $i<j$, is chosen uniformly at random, and the state jumps from vertex x to vertex $\pi_{i,j}x$. For any given $1 \leq \ell \leq N$, the number of pairs $i<j$ containing ℓ is $N-1$, and hence the fraction of the jumps that change the state of the ℓth particle is $2/N$.

In order to have that all particles update with an expected time of order 1, independent of N, we therefore multiply the Dirichlet form by N, to obtain a family of processes, indexed by N, in which the expected waiting times for updates of each particle are of order 1, independent of N. This is physically motivated, but as we shall see, it is also convenient for the induction.

Definition

Define the Dirichlet form

$$\begin{aligned}\mathcal{D}_{N,\mathbf{k}}(f,f) &= \frac{N}{2}\binom{N}{2}^{-1}\sum_{x\in\mathcal{V}_{N,\mathbf{k}}}\sum_{i<j}(f(\pi_{i,j}x)-f(x))^2\mu_{N,\mathbf{k}}(x)\\ &= \frac{1}{N-1}\sum_{x\in\mathcal{V}_{N,\mathbf{k}}}\sum_{i<j}(f(\pi_{i,j}x)-f(x))^2\mu_{N,\mathbf{k}}(x)\end{aligned}$$

where the pair permutations $\pi_{i,j}$ acts on x by swapping the i and jth entries. Also define $\Delta_{N,\mathbf{K}}$ to be the spectral gap associated to this Dirichlet from. That is

$$\Delta_{N,\mathbf{k}} = \inf\left\{\mathcal{D}_{N,k}(f,f)\ :\ \|f\|_{L^2(\mu_{N,\mathbf{k}})} = 1\ ,\ \langle f,1\rangle_{L^2(\mu_{N,\mathbf{k}})} = 0\ .\right\}$$

Remark: Comparing this with with the Dirichlet form of the graph Laplacian $L_{N,\mathbf{k}}$, we see that its gap, $\Gamma_{N,\mathbf{k}}$, and $\Delta_{N,\mathbf{k}}$ are related by

$$\Delta_{N,\mathbf{k}} = \frac{2}{N-1}\Gamma_{N,\mathbf{k}} \ .$$

Remark: We need only consider graphs $\mathcal{G}_{N,\mathbf{k}}$ where $\mathbf{k} = (k_0, \dots, k_{r-1})$ is such that

$$k_m \geq 1 \quad \text{for each} \quad 0 \leq m \leq r-1 \ .$$

Then $\mathcal{G}_{N,\mathbf{k}}$ is a graph for N particles that truly have r different energy levels. If it were the case that $k_m = 0$ for some m, the energy e_m would play no role, and the graph would be identical to another graph with a reduced set of $r' < r$ energy levels.

There is a bijection of $\mathcal{V}_{N,\mathbf{k}}$ with a union of vertex sets of graphs for $N-1$ particles: For $0 \leq m \leq r-1$, define $\mathbf{k}^{(m)}$ to be obtained from $\mathbf{k}$ by replacing k_m with $k_m - 1$. For each $1 \leq \ell \leq N$ we define a map

$$T_\ell : \left(\bigcup_{m=0}^{r-1} \mathcal{V}_{N-1,\mathbf{k}^{(m)}} \right) \to \mathcal{V}_{N,\mathbf{k}}$$

by

$$T_\ell(x) = (x_1, \dots, x_{\ell-1}, e_m, x_\ell, \dots, x_{N-1}) \quad \text{for} \quad x \in \mathcal{V}_{N-1,\mathbf{k}^{(m)}}$$

with the obvious modifications for $\ell = 1$ or $\ell = N-1$.

Lemma

Let $\{e_0, \ldots, e_{r-1}\}$ *be given along with* $\mathbf{k} = (k_0, \ldots, k_{r-1})$ *where each* k_m *is a strictly positive integer and* $\sum_{m=1} k_m = N$*. Then the spectral gap* $\Delta_{N,\mathbf{k}}$ *satisfies*

$$\Delta_{N,\mathbf{k}} \geq \frac{N(N-2)}{(N-1)^2} \min\{\Delta_{N-1,\mathbf{k}^{(m)}} \ : \ 0 \leq m \leq r-1\}$$

The key is the identity

$$\binom{N}{2}^{-1} \sum_{i<j} (f(\pi_{i,j}x) - f(x))^2 =$$

$$\frac{1}{N} \sum_{\ell=1}^{N} \left(\binom{N-1}{2}^{-1} \sum_{i<j, i,j \neq \ell} (f(\pi_{i,j}x) - f(x))^2 \right) ,$$

For each $1 \leq \ell \leq N$, and each $m \in \{0, \ldots, r-1\}$, define

$$\mathcal{D}^{\ell,m}_{N,\mathbf{k}}(f,f) =$$

$$\frac{1}{N-2} \sum_{x \in \mathcal{V}_{N,\mathbf{k}}, x_\ell = e_m} \left(\sum_{i<j, i,j \neq \ell} (f(\pi_{i,j}x) - f(x))^2 \right) \mu_{N-1,\mathbf{k}^{(m)}} \ .$$

Using this, the key identity, and

$$\mu_{N,\mathbf{k}} = \sum_{m=0}^{r-1} \frac{k_m}{N} \mu_{N-1,\mathbf{k}^{(m)}} \ ,$$

we obtain:

$$\mathcal{D}_{N,\mathbf{k}}(f,f) = \frac{1}{N}\sum_{\ell=1}^{N} \frac{N}{N-1}\sum_{m=0}^{r-1} \mathcal{D}^{\ell,m}_{N,\mathbf{k}}(f,f)\frac{k_m}{N}$$

Now the issue is that if f is orthogonal to the constant on $\mathcal{V}_{N,\mathbf{k}}$, its restriction to each $\mathcal{V}_{N-1,\mathbf{k}^{(m)}}$ need not have this property. Hence we define the operator P_ℓ on $L^2(\mathcal{V}_{N,\mathbf{k}})$ as follows: On the set $\{x \ : \ x_\ell = e_m\}$,

$$P_\ell f(x) := \mu_{N,\mathbf{k}^{(m)}} \sum_{y\in\mathcal{V}_{N,\mathbf{k}} \ : \ y_\ell = e_m} f(y) \ .$$

Note that

$$\mathcal{D}^{\ell,m}_{N,\mathbf{k}}(f,f) = \mathcal{D}^{\ell,m}_{N,\mathbf{k}}(f - P_\ell f, f - P_\ell f)$$

$$\mathcal{D}_{N,\mathbf{k}}^{\ell,m}(f - P_\ell f, f - P_\ell f) \geq \Delta_{N-1,\mathbf{k}^{(m)}} \|f - P_\ell f\|^2_{L^2(\mu_{N-1,\mathbf{k}^{(m)}})}$$

$$\begin{aligned} \sum_{m=1}^{r-1} \|f - P_\ell f\|^2_{L^2(\mu_{N-1,\mathbf{k}^{(m)}})} \frac{k_m}{N} &= \|f - P_\ell f\|_{L^2(\mu_{N,\mathbf{k}})} \\ &= \|f\|^2_{L^2(\mu_{N,\mathbf{k}})} - \langle f, P_\ell f \rangle_{L^2(\mu_{N,\mathbf{k}})} \ . \end{aligned}$$

Then taking f to be a normalized gap eigenfunction for $\Delta_{N,\mathbf{k}}$,

$$\Delta_{N,\mathbf{k}} \geq \min\left\{\Delta_{N-1,\mathbf{k}^{(m)}} \ : \ 0 \leq m \leq r-1\right\} \frac{N}{N-1}(1 - \langle f, Pf \rangle_{L^2(\mu_{N,\mathbf{k}})})$$

where

$$P = \frac{1}{N} \sum_{\ell=1}^{N} P_\ell \ .$$

Definition

Let $N \geq 3$ and let $\lambda_{N,\mathbf{k}}$ denote the second largest eigenvalue of P

$$\lambda_{N,\mathbf{k}} = \sup\left\{ \langle h, Ph\rangle_{L^2(\mu_{N,\mathbf{k}})} \ : \ \|h\|_{L^2(\mu_{N,\mathbf{k}})} = 1 \ , \ \langle h, 1\rangle_{L^2(\mu_{N,\mathbf{k}})} = 0 \right\} \ .$$

Finally we have

$$\Delta_{N,\mathbf{k}} \geq \min\left\{ \Delta_{N-1,\mathbf{k}^{(m)}} \ : \ 0 \leq m \leq r-1 \right\} \frac{N}{N-1}(1 - \lambda_{N,\mathbf{k}}) \ .$$

Lemma

Let $N \geq 3$*. The spectrum of* P *is the set* $\{0, 1/(N-1), 1\}$*. In particular,*

$$\lambda_{N,\mathbf{k}} = \frac{1}{N-1} \ .$$

Moreover, the eigenspace corresponding to 1 *consists of the constant functions, and the eigenspace corresponding to* $1/(N-1)$ *has dimension* $(r-1)(N-1)$ *and a basis for it is the set of functions of the form*

$$f_{m,\ell}(x) = g_m(x_\ell) \quad 1 \leq m \leq r-1 \quad \text{and} \quad 1 \leq \ell \leq N-1$$

where $\{g_1, \dots, g_{r-1}\}$ *is a basis for* $\mathcal{K}_{N,\mathbf{k}}$*.*

Sketch: Suppose that f is an eigenfunction of P with an eigenvalue $0 < \lambda < 1$. Then necessarily f is orthogonal to the constants and

$$\lambda f(x) = Pf(x) = \frac{1}{N}\sum_{\ell=1}^{N} g_\ell(x_\ell) \qquad \text{where} \qquad g_\ell(x_\ell) = P_\ell f(x)\ .$$

Therefore, any such eigenfunction f has the very special form

$$f(x) = \frac{1}{\lambda N}\sum_{\ell=1}^{N} g_\ell(x_\ell)\ .$$

$0 = \sum_{x\in\mathcal{V}_{N,\mathbf{k}}} f(x)\mu_{N,\mathbf{k}}(x) = \sum_{\ell=1}^{N}\langle g_\ell, 1\rangle_{L^2(\nu_{N,\mathbf{k}})}$, so we may assume each g_ℓ is orthogonal to the constants. Apply P_n to both sides of the last displayed equation. This leads to a further reduction.

Let $\nu_{N,\mathbf{k}}$ denote the probability measure on $\{e_0, \dots, e_{r-1}\}$ given by

$$\nu_{N,\mathbf{k}}(\{e_m\}) = \frac{k_m}{N} \ .$$

Definition

Define the self-adjoint K operator on $L^2(\nu_{N,\mathbf{k}})$ in terms of its associated quadratic from as follows:

$$\langle g, Kh \rangle_{L^2(\nu_{N,\mathbf{k}})} := \sum_{x \in \mathcal{V}_{N,\mathbf{k}}} g(x_1)h(x_N)\mu_{N,\mathbf{k}}(x) \ .$$

Lemma

The spectrum of the K on $L^2(\nu_{N,\mathbf{k}})$ is $\{1, -1/(N-1)\}$. The eigenspace corresponding to the eigenvalue 1 consists of the constant functions on $\{e_0, \dots, e_{r-1}\}$, and the eigenspace corresponding to the eigenvalue $-1/(N-1)$ is the space $\mathcal{K}_{N,\mathbf{k}}$.

Proof:

$$\sum_{x \in \mathcal{V}_{N,\mathbf{k}}} g(x_1)h(x_N)\mu_{N,\mathbf{k}}(x) = \frac{1}{N(N-1)} \sum_{m=0}^{r-1} g(e_m)h(e_m)k_m(k_m - 1) + \frac{1}{N(N-1)} \sum_{m \neq n} g(e_m)h(e_n)k_m k_n \ .$$

From here it follows easily that $Kh(e_m) = \sum_{n=0}^{r-1} K_{m,n}h(e_n)$ where

$$(N-1)K_{m,n} = \begin{cases} k_n - 1 & n = m \\ k_n & n \neq m \end{cases} \ .$$

□

Applying P_n to

$$f(x) = \frac{1}{\lambda N} \sum_{\ell=1}^{N} g_\ell(x_\ell) \ ,$$

yields

$$(N\lambda - 1) g_n = \sum_{\ell \neq n} K g_\ell \ .$$

Define M to be the $N \times N$ matrix with $M_{n,\ell} = \begin{cases} 0 & n = \ell \\ 1 & n \neq \ell \end{cases}$. Then

$$(N\lambda - 1)\vec{g} = M \otimes K\vec{g} \ .$$

Form here it is easy to see that the only possible value for λ, given that $\lambda \neq 0, 1$, is $\lambda = \frac{1}{N-1}$.

Knowing this,

$$\Delta_{N,\mathbf{k}} \geq \min\left\{\Delta_{N-1,\mathbf{k}^{(m)}} \ : \ 0 \leq m \leq r-1\right\} \frac{N}{N-1}(1-\lambda_{N,\mathbf{k}})$$

becomes

$$\Delta_{N,\mathbf{k}} \geq \min\left\{\Delta_{N-1,\mathbf{k}^{(m)}} \ : \ 0 \leq m \leq r-1\right\} \frac{N(N-2)}{(N-1)^2} \ .$$

We now inductively prove

$$\Gamma_{N,\mathbf{k}} = N \qquad \text{and} \qquad \Delta_{N,\mathbf{k}} = \frac{2N}{N-1}$$

Consider $N = 2$. The only non-trivial choice for $\mathbf{k}$ is with $r = 2$ and $\mathbf{k} = (1, 1)$. There are two vertices (e_0, e_1) and (e_1, e_0) and the single edge connects them. This is a complete graph, and hence $\Gamma_{2,(1,1)} = 2$ and $\Delta_{2,(1,1)} = 4$. In summary, for $N = 2$, the is only one non-trivial choice of $\mathbf{k}$, and for this choice, $\Delta_{2,\mathbf{k}} = 4$.

$$\Delta_{N,\mathbf{k}} \geq \frac{N(N-2)}{(N-1)^2}\frac{2(N-1)}{N-2} = \frac{2N}{N-1} \ .$$

Also, if f is not a gap eigenfunction of P, it cannot be a gap eigenfunction of f because there would be strict inequality in the induction. And we know that each gap eigenfucntion of P is a gap eigenfunction of the Laplacian. $\square$.

Thank you for your Interest!