Recent applications concerning WZ theory

John M. Campbell

Dalhousie University

Thu., Nov. 16, 2023, 5:00pm

Recent applications concerning WZ theory

- **Numerical analysis**
  - Series accelerations
  - Improved numerical algorithms for the calculation of universal constants
- **Computer algebra**
  - Applications toward practical problems associated with current CAS software
- **Special functions**
  - Generalizations of and new techniques related to Ramanujan's hypergeometric series
- **Combinatorics**
  - New techniques that may be applied to finite sums that naturally arise within combinatorics but that are inevaluable with current CAS software

A practical problem

J. M. Campbell, Nested radicals obtained via the Wilf–Zeilberger method and related results, *Maple Trans.* **3** no. 3 (2023), Article 16011.

Numerically computing $\sqrt[r]{z}$ or $\sqrt[r]{q}$ is straightforward for $z \in \mathbb{Z}$ and $q \in \mathbb{Q}$.

The situation becomes much more difficult for *nested* radicals.

$$10\sqrt{2-\sqrt{2}} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \begin{bmatrix} -\frac{5}{8}, -\frac{3}{8}, \frac{3}{8}, \frac{5}{8} \\ \frac{1}{2}, \frac{1}{2}, 1, \frac{3}{2} \end{bmatrix}_n \left(64n^2+5\right)$$

A practical problem

How could we obtain fast-converging, rational, hypergeometric series for nested radicals such as $\sqrt{2-\sqrt{2}}$?

The repeated differentiation of compositions such as $\sqrt{2-\sqrt{1+x}}$ becomes more and more unwieldy.

Similarly, the generalized binomial theorem would not provide a rational expansion, and similarly for expansions of trigonometric expressions according to $\sin\left(\frac{\pi}{8}\right) = \frac{\sqrt{2-\sqrt{2}}}{2}$.

$$\frac{14\sqrt{2+\sqrt{2}}}{3} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \begin{bmatrix} -\frac{7}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{7}{8} \\ \frac{1}{2}, \frac{1}{2}, 1, \frac{3}{2} \end{bmatrix}_n \left(192n^2+7\right)$$

Ramanujan-inspired series

Ramamnujan's series of convergence rate $\frac{1}{4}$:

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1 \end{matrix}\right]_n (6n+1),$$

$$\frac{8}{\pi} = \sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n \left[\begin{matrix} \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \\ 1, 1, 1 \end{matrix}\right]_n (20n+3).$$

New formulas of the same convergence rate:

$$84\sqrt{2-\sqrt{2+\sqrt{2}}} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \left[\begin{matrix} -\frac{9}{16}, -\frac{7}{16}, \frac{7}{16}, \frac{9}{16} \\ \frac{1}{2}, \frac{1}{2}, 1, \frac{3}{2} \end{matrix}\right]_n \left(256n^2+21\right),$$

$$64\sqrt{2+\sqrt{2+\sqrt{2}}} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \left[\begin{matrix} -\frac{1}{16}, \frac{1}{16}, \frac{15}{16}, \frac{17}{16} \\ \frac{1}{2}, 1, \frac{3}{2}, \frac{3}{2} \end{matrix}\right]_n \left(768n^2+512n+127\right).$$

A WZ-based series acceleration method

$$F(n+1,k)-F(n,k)=G(n,k+1)-G(n,k)$$

Guillera's trick:

$$\sum_{n=0}^{\infty}(G(n,k+1)-G(n,k))=\lim_{n\to\infty}F(n+1,k)-F(0,k)$$

Let (F,G) be a WZ pair, and suppose that

$$\lim_{n\to\infty}F(n+1,k)=0$$

for all k. From Guillera's trick, we obtain that

$$-F(0,k)=\sum_{n=0}^{\infty}(G(n,k+1)-G(n,k)).$$

A WZ-based series acceleration method

We set the variable k as b, and then as $b+1$, and then as $b+2$, and so forth. Adding the identities that result from this, a telescoping argument gives us that

$$-\sum_{n=0}^{m} F(0, b+n) = \sum_{n=0}^{\infty} (G(n, b+m+1) - G(n, b)).$$

If possible, we would want to argue that $\lim_{m\to\infty} \sum_{n=0}^{\infty} G(n, b+m+1)$ may be simplified in closed form:

$$-\sum_{n=0}^{\infty} F(0, b+n) = \text{constant} - \sum_{n=0}^{\infty} G(n, b).$$

We apply a "shifted" version of the above acceleration method, starting with the variant

$$F(n+p+1, k) - F(n+p, k) = G(n+p, k+1) - G(n+p, k)$$

for a free parameter p.

Series acceleration identities

Gauss's hypergeometric theorem:

$$ {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, 1\right] = \Gamma\left[\begin{matrix} c, c-a-b \\ c-a, c-b \end{matrix}\right]. $$

$$ F(n,k) = \frac{(c+n-1)!(c+n)!(-n-1)_k(-n)_k}{(c-1)!k!(c+2n)!(c)_k} $$

$$ R(n,k) = \frac{k(c+k-1)\left(ck-2cn-3c+2kn+2k-3n^2-7n-4\right)}{(c+2n+1)(c+2n+2)(k-n-2)(k-n-1)}, $$

$$ G(n,k) = F(n,k)R(n,k) $$

Series acceleration identities

Theorem

For $G(n,k)$ as specified,

$$-\Gamma\left[\begin{matrix} k-p-1, k-p, c+p, c+p+1 \\ k+1, c+k, -p-1, -p, c+2p+1 \end{matrix}\right] {}_3F_2\left[\left.\begin{matrix} 1, k-p-1, k-p \\ k+1, c+k \end{matrix}\,\right|\, 1\right]$$

equals

$$\lim_{m\to\infty} \sum_{n=0}^{\infty} G(n+p, k+m) - \sum_{n=0}^{\infty} G(n+p, k),$$

if the above limit exists and all of the above sums are convergent.

Series acceleration identities

$$\pi^2 = \frac{128}{156279375} \sum_{n=0}^{\infty} \left(-\frac{1}{27}\right)^n \left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, 1, 1, \frac{3}{2}, \frac{3}{2}, 2, \frac{5}{2} \\ \frac{7}{4}, \frac{11}{6}, \frac{13}{6}, \frac{9}{4}, \frac{9}{4}, \frac{11}{4}, \frac{11}{4}, \frac{13}{4} \end{matrix} \right]_n \cdot (1605632n^8 + 17633280n^7 + 83231232n^6 + 220523520n^5 + 358672608n^4 + 366633840n^3 + 229955938n^2 + 80885565n + 12211200).$$

$$\sqrt[3]{2} = \frac{1}{2952069120} \sum_{n=0}^{\infty} \left(-\frac{1}{27}\right)^n \left[\begin{matrix} \frac{5}{12}, \frac{2}{3}, \frac{11}{12}, \frac{11}{12}, \frac{7}{6}, \frac{7}{6}, \frac{17}{12}, \frac{13}{6} \\ 1, \frac{19}{12}, \frac{11}{6}, \frac{25}{12}, \frac{7}{3}, \frac{7}{3}, \frac{7}{3}, \frac{17}{6} \end{matrix} \right]_n \cdot (2550030336n^8 + 22995809280n^7 + 89131529088n^6 + 193824175104n^5 + 258432859368n^4 + 216141735204n^3 + 110617505702n^2 + 31637635223n + 3867273410).$$

Series acceleration identities

J. M. Campbell, On Guillera's ${}_7F_6\left(\frac{27}{64}\right)$-series for $1/\pi^2$, *Bull. Aust. Math. Soc.* **108** no. 3 (2023), 464–471.

How can the following series due to Guillera be generalized?

$$\frac{48}{\pi^2} = \sum_{k=0}^{\infty} \left(\frac{27}{64}\right)^k \begin{bmatrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3} \\ 1, 1, 1, 1, 1 \end{bmatrix}_k \left(74k^2 + 27k + 3\right),$$

$$\frac{16\pi^2}{3} = \sum_{k=0}^{\infty} \left(\frac{27}{64}\right)^k \begin{bmatrix} 1, 1, 1, \frac{5}{6}, \frac{7}{6} \\ \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \end{bmatrix}_k \left(74k^2 + 101k + 35\right).$$

A “strange” computer-discovered identity

S. B. Ekhad, Forty “strange” computer-discovered [and computer-proved (of course!)] hypergeometric series evaluations, (2004).

$$ {}_2F_1\left[\begin{matrix} -n, -4n-\frac{1}{2} \\ -3n \end{matrix} \,\middle|\, -1\right] = \left(\frac{64}{27}\right)^n \left[\begin{matrix} \frac{3}{8}, \frac{5}{8} \\ \frac{1}{3}, \frac{2}{3} \end{matrix}\right]_n $$

$$ F(n,k) = \frac{\pi 2^{-6n} \sec\left(\frac{\pi}{8}\right) \Gamma\left(4n+\frac{3}{2}\right) \Gamma(3n-k+1)}{\Gamma(k+1)\Gamma\left(n+\frac{3}{8}\right)\Gamma\left(n+\frac{5}{8}\right)\Gamma(n-k+1)\Gamma\left(4n-k+\frac{3}{2}\right)}. $$

New ${}_pF_q\left(\frac{27}{64}\right)$-series

J. M. Campbell, On Guillera's ${}_7F_6\left(\frac{27}{64}\right)$-series for $1/\pi^2$, *Bull. Aust. Math. Soc.* **108** no. 3 (2023), 464–471.

$$3\sqrt{2} = \sum_{k=0}^{\infty} \left(\frac{27}{64}\right)^k \begin{bmatrix} -\frac{1}{6}, -\frac{1}{8}, \frac{1}{8}, \frac{1}{6} \\ \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1 \end{bmatrix}_k \left(592k^2 - 154k + 3\right)$$

$$16\sqrt{2} = \sum_{k=0}^{\infty} \left(\frac{27}{64}\right)^k \begin{bmatrix} -\frac{1}{6}, \frac{1}{6}, \frac{3}{8}, \frac{5}{8} \\ \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2} \end{bmatrix}_k \left(1184k^3 + 876k^2 + 216k + 29\right)$$

Series acceleration identities

P. Levrie and J. Campbell, Series acceleration formulas obtained from experimentally discovered hypergeometric recursions, *Discrete Math. Theor. Comput. Sci.* **24** no. 2 (2023), dmtcs:9557.

$$h(x,y) = r_1(x,y) + r_2(x,y)\, h(x,y+1)$$

summand(x,y,k) : summand of $h(x,y)$

$$\sum_{k=0}^{\infty} \text{summand}(x,y,k) = \sum_{j=0}^{m-1} \left(\prod_{i=0}^{j-1} r_2(x,y+i) \right) r_1(x,y+j) + \left(\prod_{i=0}^{m-1} r_2(x,y+i) \right) \sum_{k=0}^{\infty} \text{summand}(x,y+m,k)$$

Generalizing Guillera's formula

Celebrated results due to Guillera include:

$$\frac{1}{\pi^2} = \frac{1}{8}\sum_{n=0}^{\infty} \left(-2^{-12}\right)^n \binom{2n}{n}^5 \left(20n^2 + 8n + 1\right)$$

$$\frac{128}{\pi^2} = \sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}^5}{2^{20n}} (820n^2 + 180n + 13)$$

These can be generalized through recursions for:

$$s(x,y) = \left(x + \frac{y-1}{2}\right) \cdot {}_5F_4\left[\begin{matrix} x, x, x, x, x + \frac{y+1}{2}, 1 \\ x+y, x+y, x+y, x+y, x+\frac{y-1}{2} \end{matrix} \,\middle|\, 1\right]$$

$$s(\tfrac{1}{2}, \tfrac{3}{2}) = 4 - \frac{32}{\pi^2}$$

Solution to a problem due to Chu and Kiliç

In a 2021 article in the *Rocky Mountain Journal of Mathematics*, Chu and Kiliç included a conjectured evaluation for

$$\sum_{k=0}^{n}(-1)^k\binom{2n-2k}{n-k}\binom{n+k}{2k}C_k,$$

and left it as an open problem to prove a closed-form evaluation for this sum.

Amazingly, current implementations of the WZ method cannot be used to obtain WZ proofs of the Chu–Kiliç formula!

Solution to a problem due to Chu and Kiliç

$$sum\left(\frac{(-1)^k \cdot \mathrm{binomial}(2\cdot n-2\cdot k, n-k)\cdot \mathrm{binomial}(n+k, 2\cdot k)\cdot \mathrm{binomial}(2\cdot k, k)}{k+1}, k=0..n\right)$$

$$\frac{2^{2n}\,\Gamma\left(n+\frac{1}{2}\right)\mathrm{hypergeom}\left([-n,-n,n+1],\left[2,-n+\frac{1}{2}\right],\frac{1}{4}\right)}{\sqrt{\pi}\,\Gamma(n+1)}$$

Chu and Kiliç (2021) experimentally discovered that it seems that for each of the residue classes for $n \bmod 6$, the binomial sum under consideration appears to reduce to a closed-form, hypergeometric expression.

Solution to a problem due to Chu and Kiliç

$$f := \frac{(-1)^k \binom{12n+2-2k}{6n+1-k} \binom{6n+1+k}{2k} \binom{2k}{k} (3n+1)(6n+1)}{(k+1)(8n+1)\binom{8n}{4n}\binom{4n}{2n}}$$

```
seq(sum(f, k = 0..6*n + 1), n = 0..10)
                                        1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1

with(SumTools[Hypergeometric]) :
r := 1
                                                                 r := 1

WZpair := WZMethod(f, r, n, k, 'cert') :
Error, (in SumTools:-Hypergeometric:-WZMethod) WZ method fails
```

Solution to a problem due to Chilç and Kiliç

$WZpair := WZMethod(f, r, n, k, 'cert')$:

Error, (in SumTools:-Hypergeometric:-WZMethod) WZ method fails

"Why" does the Maple implementation of the WZ method fail, here?

In theory, there "should" be a WZ proof certificate for the finite hypergeometric sum that evaluates to a constant.

How could the Maple implementation of the WZ method be improved or corrected to deal with this problem? What's going wrong, here?

Solution to a problem due to Chu and Kiliç

```
WZpair := WZMethod(f, r, n, k, 'cert') :
Error, (in SumTools:-Hypergeometric:-WZMethod) WZ method fails
```

Interestingly, the Maple implementation of *Zeilberger's algorithm*, as opposed to the WZ method, can be used to formulate a full solution to the problem proposed by Chu and Kiliç.

J. M. Campbell, Solution to a problem due to Chu and Kiliç, *Integers* **22** (2022), #A46.

A "cubic" WZ method

J. M. Campbell, A WZ proof for a Ramanujan-like series involving cubed binomial coefficients, *J. Difference Equ. Appl.* **27** no. 10 (2021), 1507–1511.

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$$

$$\sum_{k=0}^{\infty} (F(n+1,k) - F(n,k))^3 =$$

$$-3 \sum_{k=0}^{\infty} \left(G^2(n,k+1)G(n,k) - G(n,k+1)G^2(n,k) \right)$$

A "cubic" WZ method

Set $F(n,k) = \frac{\binom{n}{k}}{2^n}$ and $G(n,k) = -\frac{\binom{n}{k-1}}{2^{n+1}}$. Differentiate the "cubic" WZ identity and set $n = -\frac{1}{2}$.

$$\sum_{k=0}^{\infty} \frac{C_k^3\, O_k(4k+3)\left(4k^2+6k+3\right)}{(-64)^k} =$$

$$8 - \frac{8}{\pi} - \frac{64\sqrt{2}\pi}{\Gamma^2\left(\frac{1}{8}\right)\Gamma^2\left(\frac{3}{8}\right)} - \frac{3\Gamma^2\left(\frac{1}{8}\right)\Gamma^2\left(\frac{3}{8}\right)}{4\sqrt{2}\pi^3}$$

This recalls Guillera's evaluation of the Ramanujan-like series

$$\sum_{n=0}^{\infty}\left(-\frac{1}{64}\right)^n \binom{2n}{n}^3 (4n+1)H_n = \frac{1}{6\sqrt{2}\pi}\left(\frac{\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}\right)^2 - \frac{4\ln(2)}{\pi}.$$

WZ proofs of Ramanujan's series

J. M. Campbell and P. Levrie, Further WZ-based methods for proving and generalizing Ramanujan's series, *J. Difference Equ. Appl.* **29** no. 3 (2023), 366–376.

$$\frac{16}{\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{12n} \binom{2n}{n}^3 (42n+5)$$

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{8n} \binom{2n}{n}^3 (6n+1)$$

Writing $v(x) = \sum_{n=1}^{\infty} \frac{(x)_n^2}{(2x+1)_n^2}$,

$$v(x) = \frac{(21x+13)x^2}{2^3(2x+1)^3} + \frac{x^2(x+1)}{2^3(2x+1)^3} v(x+1).$$

WZ proofs of Ramanujan's series

$v\left(\frac{1}{2}\right) = \frac{16}{\pi} - 5$ is immediate from a classical formula of Forsyth.
The recursion for v may be proved through the WZ method, and repeated applications of this recursion has the effect of accelerating Forsyth's series.
This gives us a completely different WZ proof compared to that in Guillera's seminal 2002 article, which relied on

$$G(n,k) = \frac{(-1)^n(-1)^k}{2^{10n}2^{2k}}(20n+2k+3)\frac{\binom{2k}{k}^2\binom{2n}{n}^2\binom{4n-2k}{2n-k}}{\binom{2n}{k}\binom{n+k}{n}}$$

and

$$\sum_{n=0}^{\infty} G(n,0) = \sum_{n=0}^{\infty} H(n,0),$$

where $H(n,k) = F(n+1, n+k) + G(n, n+k)$.

Zeilberger's computer proof of Ramanujan's formula

S. B. Ekhad and D. Zeilberger, A WZ proof of Ramanujan's formula for π, in *Geometry, Analysis and Mechanics*, World Sci. Publ., River Edge, NJ, 1994, pp. 107–108.

$$\frac{\Gamma\left(n+\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma(n+1)} = \sum_{k=0}^{n} (-1)^k (4k+1) \frac{\left(\frac{1}{2}\right)_k^2 (-n)_k}{(k!)^2 \left(n+\frac{3}{2}\right)_k}$$

Zeilberger used a WZ proof of the above identity to prove Ramanujan's famous formula

$$\frac{2}{\pi} = \sum_{n=0}^{\infty} \left(-\frac{1}{64}\right)^n \binom{2n}{n}^3 (4n+1).$$

Zeilberger's computer proof of Ramanujan's formula

How can we obtain similar results, using variants of the below identity used by Zeilberger?

$$\frac{\Gamma\left(n+\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma(n+1)} = \sum_{k=0}^{n}(-1)^k(4k+1)\frac{\left(\frac{1}{2}\right)_k^2(-n)_k}{(k!)^2\left(n+\frac{3}{2}\right)_k}$$

By removing the $(-1)^k$ factor, and by again applying the WZ method with respect to the resultant sum, we can prove

$$\sum_{k=0}^{\infty}\left(\frac{1}{16}\right)^k\binom{2k}{k}^2\frac{1}{4k+3} = \frac{2}{\pi} - \frac{2\Gamma^2\left(\frac{3}{4}\right)}{\Gamma^2\left(\frac{1}{4}\right)},$$

and a similar approach can be used to prove a similar formula due to Ramanujan.

Physics-based applications of WZ theory

$H_n(x)$: The n^{th} Hermite polynomial
The eigenfunctions of the quantum harmonic oscillator:

$$\psi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{-x^2/2} H_n(x)$$

$$\psi_{2n}(0) = \frac{H_{2n}(0)}{\sqrt{2^{2n}(2n)! \sqrt{\pi}}}$$

Fassari et al. (2021) explored how the series

$$S = \frac{1}{2} \sum_{n=1}^{\infty} \frac{\psi_{2n}^4(0)}{n}$$

may be applied in relation to the quantum harmonic oscillator

$$S = \frac{1}{2} \sum_{n=1}^{\infty} \frac{\psi_{2n}^4(0)}{2n} = \frac{2}{\pi^2} (\pi \ln 2 - 2G)$$

Physics-based applications of WZ theory

J. M. Campbell, On the even-indexed eigenfunctions of the quantum harmonic oscillator, *Rep. Math. Phys.* **92** no. 2 (2023), 197–207.

By the Dixon summation theorem,

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(n-k)!(n+k+1)!} = \left(\frac{4^n n!}{(2n+1)!} \right)^2 .$$

$$F(n,k) = \frac{\text{summand}}{\text{rhs}}$$

$$\sum_{n=0}^{\infty} (F(n+1,r) - F(n,r)) = \sum_{n=0}^{\infty} G(n,r+1) - \sum_{n=0}^{\infty} G(n,r)$$

$$\sum_{n=0}^{\infty} (F(n+1,r+1) - F(n,r+1)) = \sum_{n=0}^{\infty} G(n,r+2) - \sum_{n=0}^{\infty} G(n,r+1)$$

Under Review

How can we obtain *two-term* recurrences

$$p_1(n)F(n+r,k)+p_2(n)F(n,k)=G(n,k+1)-G(n,k)$$

from Zeilberger's algorithm, and in such a way so as to obtain series accelerations, using the same or a similar recursive approach as before?

$$\frac{279936}{5\pi}=\sum_{j=0}^{\infty}\left(\frac{4}{27}\right)^{j}\begin{bmatrix}\frac{1}{6},\frac{7}{12},\frac{5}{6},\frac{11}{12},\frac{13}{12},\frac{17}{12}\\ 1,\frac{4}{3},\frac{3}{2},\frac{5}{3},2,2\end{bmatrix}_{j}\cdot (59616j^4+184032j^3+206748j^2+99324j+17017).$$

Under Review

$$3360\sqrt[3]{2} = \sum_{j=0}^{\infty} \left(\frac{4}{27}\right)^j \begin{bmatrix} \frac{7}{12}, \frac{2}{3}, \frac{5}{6}, \frac{13}{12}, \frac{7}{6}, \frac{4}{3} \\ \frac{1}{2}, 1, \frac{13}{9}, \frac{3}{2}, \frac{16}{9}, \frac{19}{9} \end{bmatrix}_j \cdot (14904j^4 + 38772j^3 + 34434j^2 + 12039j + 1330).$$

$$\frac{17414258688}{77\pi} = \sum_{j=0}^{\infty} \left(\frac{27}{256}\right)^j \begin{bmatrix} \frac{1}{6}, \frac{13}{18}, \frac{5}{6}, \frac{17}{18}, \frac{19}{18}, \frac{23}{18}, \frac{25}{18}, \frac{29}{18} \\ 1, \frac{3}{2}, \frac{5}{3}, \frac{7}{4}, 2, 2, \frac{9}{4}, \frac{7}{3} \end{bmatrix}_j (96158016j^6 + 566590464j^5 + 1373338800j^4 + 1751461056j^3 + 1238515308j^2 + 459977904j + 70018325).$$