Finding the jewel in the lotus

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The next issue of the *LMS Newsletter* will have an article giving more detail on the project.

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This graph was used by Brauer and Fowler in a very important paper in 1955, perhaps the first step to the Classification of Finite Simple Groups. They showed that there are only finitely many simple groups (of even order) containing a prescribed involution centraliser.
However, back in the dark ages of 1955, graph theory was not really a subject, so the word "graph" does not occur in their paper.

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The enhanced power graph, or cyclic graph, has x and y joined if there exists z such that both x and y are powers of z. Thus, $x \sim y$ if the group generated by x and y is cyclic.

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For example, with V. V. Swathi and M. S. Sunitha, I showed that these two graphs have the same matching number. The proof is a standard alternating chains argument.

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There is also a lot of work in particular graphs defined on particular groups, calculating their properties (chromatic number, spectrum, etc.) But I am not so interested in this.

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These groups were studied by Higman in the 1950s (who determined the solvable ones) and Suzuki in the 1960s (who found the simple ones). The complete classification was given in a little-known paper by Brandl in 1981, not using the Classification of Finite Simple Groups.

What makes a graph interesting?

Different people will give different answers to this question. As a group theorist, I prefer my graphs to have lots of symmetry. But there are other things one might ask for: various kinds of regularity such as distance-regularity; large girth for the number of vertices and edges; etc.

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What is going on?

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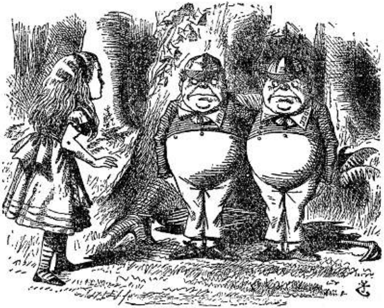
That is what I have been looking for.

Twins



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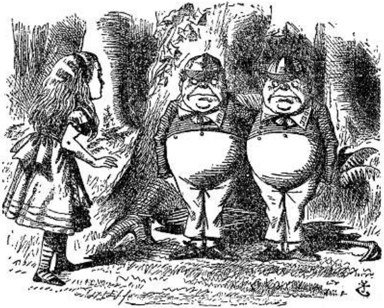

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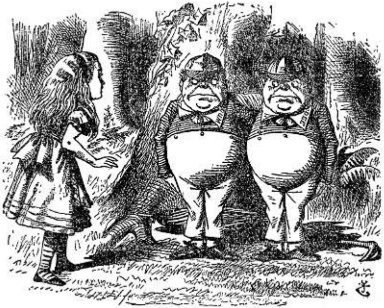

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Two vertices x, y in a graph are twins if they have the same neighbours, except possibly for one another.
So there are two kinds of twins: open twins (x not joined to y, same open neighbourhoods) and closed twins (x joined to y, same closed neighbourhoods).

Twins and automorphisms

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But these are not *interesting* automorphisms!

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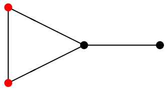

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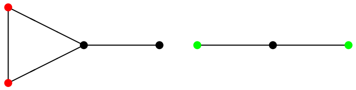

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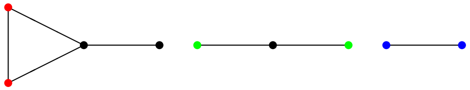

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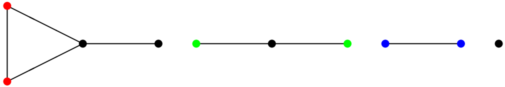

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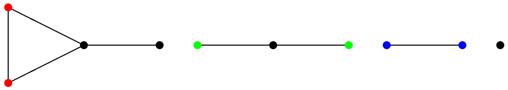

But it is not hard to show that, if we continue until no further twins remain, the graph we get is (up to isomorphism) independent of the reduction process.

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This class of graphs has been rediscovered many times, and given several different names.

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Here are some of the things we found. The first few were expected.

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- $\mathrm{PSL}(2, q)$, for $q = 4, 7, 8, 9, 17$;
- $\mathrm{Sz}(q)$, for $q = 8, 32$;
- $\mathrm{PSL}(3, 4)$.

Nothing further to say about these.

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These conditions hold for $q = 2^d$ where $d = 1, 2, 3, 4, 5, 7, 11, 13,$ $17, 19, 23, 31, 61, 101, 127, 167$.

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These conditions hold for $q = 2^d$ where $d = 1, 2, 3, 4, 5, 7, 11, 13, 17, 19, 23, 31, 61, 101, 127, 167$.
Are there infinitely many? Nobody knows!

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In particular, it is a mystery why these two groups give isomorphic connected components.
But in any case, we take the view that in these cases twin reduction has not been sufficient to blow the rubbish away.

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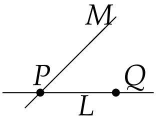

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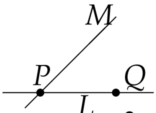

- It is semiregular, with valencies q^2 and $q + 1$ in the two blocks.

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Allowing almost simple groups, we find for example the symmetric group S_7, where we have one component with 322 vertices and seven with 35 vertices.

That's not all

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We really need to understand twin reduction better, especially in graphs on groups.

What next?

There is plenty more to explore; other types of graphs, other types of groups, etc. If you are interested in this, please try your hand. Here are some references:

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- Sucharita Biswas, Peter J. Cameron, Angsuman Das and Hiranya Kishore Dey, On difference of enhanced power graph and power graph in a finite group, *J. Combinatorial Theory* (A), **208** (2024), 105932; ; doi: `10.1016/j.jcta.2024.105932`
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… for your attention.