Searching for sequences: Irrationality beyond Apery

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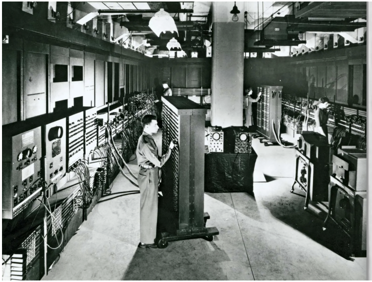
Why Grothendieck and computers are best of friends




A criteria for irrationality

Rational numbers are poorly approximated by rational numbers different from themselves.

If $\alpha = a/b \in \mathbf{Q}$, and $\alpha \neq p/q$;

$$\left|\alpha - \frac{p}{q}\right| = \left|\frac{a}{b} - \frac{p}{q}\right| = \left|\frac{aq - bp}{bq}\right| \geq \frac{1}{bq}.$$

If $\alpha \in \mathbf{R}$ and there exists a sequence $p/q \in \mathbf{Q}$ with $q \to \infty$ and

$$0 < \left|\alpha - \frac{p}{q}\right| = o\left(\frac{1}{q}\right)$$

then $\alpha \notin \mathbf{Q}$.

$$\text{Dirichet} : \alpha \in \mathbf{R} \setminus \mathbf{Q},\ \exists\ p, q \gg 1 \text{ with } 0 < \left|\alpha - \frac{p}{q}\right| \leq \frac{1}{q^2}.$$

A tricky example

Let

$$I(n) = \int_0^1 \left(\frac{t(1-t)}{(1+t)} \right)^n \frac{dt}{1+t}$$

If you expand out the numerator, you can write this as an integral combination of the following integrals:

$$\int_0^1 (1+t)^{k-1} dt \in \frac{\mathbf{Z}}{2^n k}, k = -n, \dots, -1, 1, \dots, n$$

$$\int_0^1 \frac{1}{1+t} dt = \log 2,$$

$$\text{Hence } I(n) \in \mathbf{Z} \log 2 + \frac{\mathbf{Z}}{2^n [1, 2, \dots, n]}.$$

$$\text{Actually } I(n) \in \mathbf{Z} \log 2 + \frac{\mathbf{Z}}{[1, 2, \dots, n]}.$$

A tricky example

$$I(n) \in \mathbf{Z}\log 2 + \frac{\mathbf{Z}}{[1,2,\ldots,n]} = \frac{q_n \cdot \log 2 - p_n}{[1,2,\ldots,n]}.$$

$$0 < I(n) = \int_0^1 \left(\frac{t(1-t)}{(1+t)}\right)^n \frac{dt}{1+t} \le \max \left|\frac{t(1-t)}{(1+t)}\right|^n_{t\in[0,1]}$$
$$= (3-2\sqrt{2})^n$$

$$0 < \left|\log 2 - \frac{p_n}{q_n}\right| < \frac{1}{q_n} \cdot (3-2\sqrt{2})^n \cdot [1,2,\ldots,n]$$
$$< \frac{1}{q_n}(3-2\sqrt{2})^n \cdot e^{n(1+\varepsilon)}$$
$$\ll \frac{1}{q_n}\left(\frac{2.71829}{5.82842}\right)^n = o(1/q_n).$$

An easier example

$$A(x) = 1 + x + 2x^2 + 3x^3 + 5x^4 + \ldots = \sum F_n x^n,$$
$$B(x) = x + x^2 + 2x^3 + \ldots \sum F_{n-1} x^n.$$

$$A(x) = \frac{1}{1 - x - x^2} = \frac{1}{(1 - \phi x)(1 + \phi^{-1} x)}.$$

Claim: $\phi = \lim_{n \to \infty} F_n / F_{n-1}$ is irrational.

- We want to prove that $F_n - \phi F_{n-1}$ is small — $o(1)$ suffices.
- We want $A(x) - \phi B(x)$ to have radius of convergence $R > 1$.

$$A(x) - \phi B(x) = \frac{(1 - \phi x)}{(1 - \phi x)(1 + \phi^{-1} x)} = \frac{1}{1 + \phi^{-1} x} \notin \mathbf{C}[x]$$

has radius of convergence $|\phi| = 1.618\ldots > 1$.

A general framework

If $A(x) = \sum a_n x^n$ and $B(x) = \sum b_n x^n$ are power series with:

1. We have $a_n [1, 2, \ldots, n]^m \in \mathbf{Z}$ and $b_n [1, 2, \ldots, n]^m \in \mathbf{Z}$,
2. $P(x) = B(x) - \eta A(x)$ has radius of convergence at least R,
3. $R > e^m$,
4. $P(x)$ is not a polynomial,

Then $\eta \notin \mathbf{Q}$.

Goal is to find $A(x)$ and $B(x)$ and η with

$$\log R > m.$$

A tricky example

$$A(x) = \frac{1}{1 - x - 2x^2 + x^3} = 1 + x + 3x^2 + 4x^3 + 9x^4 + 14x^5 + \dots$$

$$A(x) = \frac{1}{(1 + 2\cos(2\pi/7)x)(1 + 2\cos(4\pi/7)x)(1 + 2\cos(8\pi/7)x)}.$$

$$A(x) = \frac{1}{(x + 0.8019\dots)(x - 2.246\dots)(x - 0.554\dots)}.$$

$A(x)$ has radius of convergence $|2\cos(8\pi/7)|^{-1} = 0.5549\dots$.

$$A(x) + 2\cos(8\pi/7)xA(x)$$

has radius of convergence $(2\cos(2\pi/7))^{-1} = 0.8019\dots$. We failed!

Conformal Radius

If $0 \in \Omega \subset \mathbf{C}$ is a simply connected open region, there exists:

$$\varphi : D(0,1) \simeq \Omega, \qquad \varphi(0) = 0, \qquad \varphi \text{ a biholomorphism}$$

$$c(\Omega) := |\varphi'(0)| \quad \text{is called the conformal radius of } \Omega$$

If $\Omega = D(0, R)$, then $\varphi(z) = Rz$ and $c(\Omega) = R$.

If $\Omega = \mathbf{C} \setminus [\beta, \infty)$, then $\varphi(z) = \dfrac{4\beta z}{(1+z)^2}$ and $c(\Omega) = 4\beta$.

Theorem: (Pólya) Let $P(x) \in \mathbf{Z}[[x]]$ analytically continue to a region Ω with $c(\Omega) > 1$. Then $P(x) \in \mathbf{Q}[x]$.

A general framework, updated

If $A(x) = \sum a_n x^n$ and $B(x) = \sum b_n x^n$ are power series with:

1. We have $a_n \in \mathbf{Z}$ and $b_n \in \mathbf{Z}$,
2. $P(x) = B(x) - \eta A(x)$ has radius of convergence at least $R > 1$,
3. $P(x) = B(x) - \eta A(x)$ is holomorphic on Ω with $c(\Omega) > 1$,
4. $P(x)$ is not a polynomial (not in $\mathbf{C}[x]$),
5. $P(x)$ is not a rational function (not in $\mathbf{C}(x)$),

Then $\eta \notin \mathbf{Q}$.

$c(\mathbf{C} \setminus [(2\cos(2\pi/7))^{-1}, \infty) = 4 \cdot 0.8019\ldots = 3.207\ldots > 1$,

BUT $A(x) + 2\cos(8\pi/7)xA(x)$ is in $\mathbf{C}(x)$, so can't apply Pólya

Making Pólya's theorem more explicit

Theorem: (Pólya) Let $P(x) \in \mathbf{Z}[[x]]$ analytically continue to $\mathbf{C} \setminus [\beta, \infty)$ with $\beta > \phi = 0.618\ldots$. Then $P(x) \in \mathbf{Q}[x, (1-x)^{-1}]$.

Proof idea: To prove that $P(x)$ on $D(0, R > 1)$ is a polynomial, consider

$$\frac{1}{2\pi i} \oint \frac{P(x)}{x^{m+1}} dx = a_m.$$

For $P(x)$ on $\mathbf{C} \setminus [\beta, \infty)$, consider

$$\frac{1}{2\pi i} \oint \frac{P(x)}{x^{m+1}} \left(\frac{1}{x^2} - \frac{1}{x} \right)^n dx = a_{m+2n} - \binom{n}{1} a_{m+2n-1} + \binom{n}{2} a_{m+2n-2} \ldots \in \mathbf{Z},$$

$$\left| \frac{1}{x^2} - \frac{1}{x} \right| < 1, \quad x \in \partial\Omega.$$

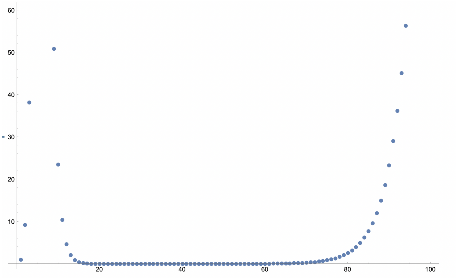
This leads to the irrationality of $2\cos(8\pi/7)$.

Non-explicit approximations

The last argument proves $2\cos(8\pi/7) \notin \mathbf{Q}$.

$$\frac{(1-x)^{10}(1-\eta x)}{1-x-2x^2+x^3} = \ldots + (521132859 - 289206918\eta)x^{27} + \ldots$$

$$|521132859 + 289206918(2\cos(8\pi/7)| = 0.0000155\ldots$$



Denominators

Here is a function with denominators and good analytic properties:

$$P(x) = -\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$$

$P(x)$ has radius of convergence 1 for each prime p.

$P(x)$ analytically continues to $\Omega = \mathbf{C} \setminus [1, \infty)$ (and more) with $c(\Omega) = 4 > e$.

$P(x)$ is transcendental so certainly not in $\mathbf{Q}(x)$.

What is the analogue of Pólya's theorem in this context?

A general framework, updated

If $A(x) = \sum a_n x^n$ and $B(x) = \sum b_n x^n$ are power series with:

1. We have $a_n[1, 2, \ldots, n]^m \in \mathbf{Z}$ and $b_n[1, 2, \ldots, n]^m \in \mathbf{Z}$,
2. $P(x) = B(x) - \eta A(x)$ has radius of convergence at least $R > e^m$,
3. There exists $\varphi : D(0, 1) \to \mathbf{C}$ with $\varphi(0) = 0$ and $P(\varphi(z))$ with $P(x) = B(x) - \eta A(x)$ holomorphic and $|\varphi'(0)| > e^m$,
4. $P(x)$ is not a polynomial,
5. $P(x)$ is not a holonomic function,

Then $\eta \notin \mathbf{Q}$.

The examples which turn up in Apéry and otherwise are ALWAYS holonomic

Need to more precisely quantify the holonomy.

Denominators

Theorem: (CDT) Arithmetic holonomy bound. Fix $m \in \mathbf{N}$, and fix $\varphi : \overline{D(0,1)} \to \mathbf{C}$ with $\varphi(0) = 0$.

Let $\mathcal{H}(\varphi, m)$ be the $\mathbf{Q}(x)$ vector space generated by

$$P(x) = \sum_{n=1}^{\infty} a_n x^n, \;\; a_n [1, 2, \ldots, n]^m \in \mathbf{Z}$$

such that $P(\varphi(x))$ is holomorphic on $D(0,1)$.

Assume that $\log |\varphi'(0)| > m$, equivalently $|\varphi'(0)| > e^m$.

$$\dim \mathcal{H}(\varphi, m) \leq \frac{\iint_{|z|=|y|=1} \log |\varphi(z) - \varphi(y)| d\mu}{\log |\varphi'(0)| - m^\flat}$$

A general framework, the grubby version

If $A(x) = \sum a_n x^n$ and $B(x) = \sum b_n x^n$ are power series with:

1. We have $a_n[1,2,\ldots,n]^m \in \mathbf{Z}$ and $b_n[1,2,\ldots,n]^m \in \mathbf{Z}$,
2. $A(x)$ and $B(x)$ are holonomic functions,
3. $P(x) = B(x) - \eta A(x)$ converges as far as the singularity $\beta \in \mathbf{C}$.

$$\begin{aligned} \text{If } \log|\beta| > m, &\quad \text{you win by Apéry.} \\ \text{If } \log|16\beta| < m, &\quad \text{you go home.} \\ \text{If } \log|16\beta| > m, &\quad \text{you come to me.} \end{aligned}$$

The constant 16 is determined from the ODE but is hard to estimate.

The cleanest scenario is $P(x)$ has singularities at $0, \alpha, \beta, \infty$ where α is very small.

If $P(x)$ extends to $\mathbf{C} \setminus [\beta, \infty)$ you get at least $\log|4\beta|$.

$L(2, \chi_{-3})$ is irrational

$$B(x) - L(2, \chi_{-3})A(x) = \sum_{n=0}^{\infty} x^n \iint_{[0,1]^2} \frac{9^n s^n t^n (1-s^3)^n (1-t^3)^n}{(1+st+s^2t^2)^{2n+1}} dsdt,$$

$A(x) \in \mathbf{Z}[[x]]$, $B(x)$ has denominators of type $[1, 2, \ldots, n]^2$.

Holonomic, singularities at 0, $\alpha = 1/9$, $\beta = 1$, and ∞.

Apery's argument would require the inequality $\log 1 > 2$.

Our starting point is $\log 16 > 2$.

We get $P(x)$, $P'(x)$, $P(x/(x-1))$, $P'(x/(x-1))$ on $\mathbf{P}^1 \setminus \{0, -1/8, 1/9, 1, \infty\}$.

Conclusion

Theorem: (CDT) Assume $L(2, \chi_{-3}) \in \mathbf{Q}$. Consider $P(x) = \sum a_n x^n$ with

1. $[1, 2, \ldots, n]^2 a_n \in \mathbf{Z}$,
2. $P(x)$ converges on $|x| < 1$.
3. $P(x)$ analytically continues on any path from 0 in $\mathbf{C} \setminus \{-1/8, 1/9, 1, 0\}$.

Then these generate a $\mathbf{Q}(x)$-vector space of dimension at most 8.

If a linear relationship exists we have four such functions. Are there others?

$$1,\ \log(1-x),\ \log(1-x)^2, \mathrm{Li}_2(x) = \sum \frac{x^n}{n^2}.$$

$${}_3F_2\left[\begin{matrix}1/2 & 1 & 1\\ 3/2 & 3/2\end{matrix}; \frac{1}{4} \cdot \left(x + \frac{x}{x-1}\right)\right] \sim \frac{1}{\sqrt{1-x}} \int \frac{\log(1-x)}{x\sqrt{1-x}} dx$$

We are done! (fine print).

Zagier's List

Case	β	α	C	c_1	c_2	L
A	8	-1	$\frac{2}{\pi\sqrt{3}}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{4}\zeta(2) = 0.4112335\ldots$
C	9	1	$\frac{3\sqrt{3}}{4\pi}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{2}L_{-3}(2) = 0.3906512\ldots$
D	ϕ^5	$-\phi^{-5}$	$\frac{\phi^{5/2}}{2\pi\sqrt[4]{5}}$	$\frac{1}{\phi\sqrt{5}}$	$\frac{\phi}{\sqrt{5}}$	$\frac{1}{5}\zeta(2) = 0.3289868\ldots$
E	8	4	$\frac{2}{\pi}$	0	1	$\frac{1}{2}L_{-4}(2) = 0.4579827\ldots$
F	9	8	$\frac{3\sqrt{3}}{\pi}$	-2	3	$\frac{5}{8}L_{-3}(2) = 0.4883140\ldots$

1. If you use **D**, $\beta = \phi^5$, you get Apéry's proof that $\zeta(2) \notin \mathbf{Q}$;
2. If you use **C**, $\beta = 1$, you get our proof;
3. If you use **E**, $\beta = 1/4$, but $\log|16\beta| - 2 < 0$.

Where to look

One wants to find ODE's $\mathcal{L} = 0$ (or holonomic sequences a_n, b_n) such that:

1. The denominators grow at most exponentially e^{nR},
2. If β is the first singularity of $P(x) = B(x) - \zeta A(x)$ then $\log|16\beta| > R$,
3. Desirable: the local monodromy is unipotent,
4. Looking at sequences of integrals $I(n)$ is too special,
5. Continued fractions are a red herring,

For example, search for $\mathcal{L}$ with degree at most 4 and at most 5 singularities all with unipotent monodromy and denominator type $[1, 2, \ldots, n]^4$.