Resurgent integer sequences

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Rutgers Experimental Mathematics Seminar, 6 February 2025

In combinatorics, we happily manipulate **formal power series**, taking no heed of whether they might **converge**. Applied mathematicians encounter series with **no** radius of convergence, about which they worry. Jean **Écalle** mediates between these communities, by telling us about resurgent **trans-series**. I shall give an account of how an **integer sequence** from a problem in **physics** exhibits **resurgence**. By way of preparation, I begin with a **simpler** integer sequence found by Pat **Devlin** and Paulina **Trifonova** in a study of **combinatorial games** played **randomly**.

1. **Asymptotics** of a **linear** recursion for **random games**
2. **Non-linear** recursion from a **Dyson-Schwinger** equation
3. **Padé–Borel** summation with **alternating** signs
4. **Trans-series** and resurgent **hyperasymptotic** expansions

Asymptotics of a linear recursion: In `https://arxiv.org/pdf/2401.16670` Pat Devlin and Paulina Trifonova identified a notable pair of **integer sequences** (α_k, β_k) defined **recursively** with $(\alpha_1, \beta_1) = (1, -1)$ and for $k > 0$

$$\alpha_{k+1} = k(4k+1)\alpha_k + \beta_k \tag{1}$$
$$\beta_{k+1} = -k(k+1)\alpha_k + (4k^2 - k - 1)\beta_k. \tag{2}$$

These **solve** the following problem. If a game of *Chomp* starts with n cells in row 1 and $k \le n$ cells in row 2, what is the **probability** in **random** play that the opening player wins? For $n \ge k > 0$ and $(n, k) \ne (1, 1)$ the answer is

$$P(n, k) = \frac{1}{2} - \frac{n\alpha_k + \beta_k}{(n+k)(n+k-1)(n+k-2)(2k-2)!}. \tag{3}$$

Conventionally, the game starts with $n = k$, where I found that

$$P(k, k) = \frac{1}{2} - \frac{(\sqrt{2}-1)S_1}{(2k-1)^{2-\mu}} + \frac{(\sqrt{2}+1)S_2}{(2k-1)^{2+\mu}} + O\left(\frac{1}{(2k-1)^{4-\mu}}\right) \tag{4}$$

at large k with

$$S_1 = \frac{2^{\mu-1}}{\Gamma(1+\mu)}, \quad S_2 = \frac{2^{-\mu-1}}{\Gamma(1-\mu)}, \quad \mu = \frac{1}{\sqrt{2}}. \tag{5}$$

Theorem 1: With $\mu = \frac{1}{2}\sqrt{2}$, $c_0 = 1$ and c_n given by the **first-order** recursion

$$2nc_n + (n-\mu)(n-1-\mu)c_{n-1} = 0 \tag{6}$$

for $n > 0$, there are **Stokes constants** (S_1, S_2) such that for large $k \geq \frac{3}{4}N$

$$\begin{aligned}\alpha_k = {} & S_1 \sum_{n=0}^{N} c_n \Gamma\left(2k - 1 + \mu - n\right) \\ & + S_2 \sum_{n=0}^{N} \overline{c}_n \Gamma\left(2k - 1 - \mu - n\right) + O\left(\overline{c}_N \Gamma(2k - N)\right)\end{aligned} \tag{7}$$

where $\overline{c}_n$ is the **conjugate** of c_n in the **quadratic number field** $\mathbb{Q}(\sqrt{2})$.

Outline of proof: Converting (1,2) to a **second-order** recursion for α_k and adopting a generic **Ansatz** $\alpha_k \sim \sum_{n\geq 0} c_n \Gamma(2k-1+\mu-n)$ we obtain, with $\nu = n - \mu$,

$$(2\nu^2 - 1)c_n + \nu(\nu-1)\left((2\nu-1)c_{n-1} + \tfrac{1}{2}(\nu-1)(\nu-2)c_{n-2}\right) = 0 \tag{8}$$

on the understanding that $c_n = 0$ for $n < 0$. At $n = 0$, this **requires** $2\mu^2 = 1$. Then (6) solves (8) for all $n > 0$. The asymptotic expansion (7) combines solutions with $\mu = \pm\frac{1}{2}\sqrt{2}$. When k and n are both large, the condition $k \geq \frac{3}{4}n$ ensures that terms in the asymptotic expansion have **decreasing** magnitudes.

Hyperasymptotics of random Chomp: In Theorem 1 the asymptotic behaviour of α_k is **governed** by $(c_n, \overline{c}_n)$ up to some **limit** imposed by the asymptotic expansion of those coefficients of asymptotic expansion. One may ask the old question: **quis custodiet ipsos custodes**, who governs those governors? If there are higher governors, who governs them? Can such a sequence of **hyperasymptotic** questions have any ending? In the present case, there is a **satisfactory** answer. The governors are governed by their **conjugates**, with

$$c_n \sim \frac{(-1)^{n+1}\mu}{\Gamma^2(1-\mu)} \sum_{m=0}^{M} 2^{m-n} \overline{c}_m \Gamma(n-2\mu-m) \tag{9}$$

$$\overline{c}_n \sim \frac{(-1)^{n}\mu}{\Gamma^2(1+\mu)} \sum_{m=0}^{M} 2^{m-n} c_m \Gamma(n+2\mu-m) \tag{10}$$

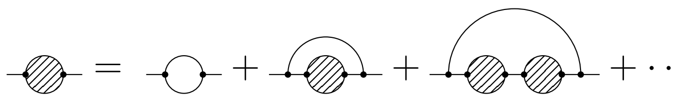
giving terms of **alternating sign** and **decreasing size** for large $n > 2M$.

The factorial growth in (9,10) accounts for the condition $k \geq \frac{3}{4}N$ in Theorem 1, where **truncation** at $n = N$ gives a **relative error** estimated by $2^{-N}\Gamma(N)\Gamma(2k-N)/\Gamma(2k)$, which becomes stationary for $N \sim \frac{4}{3}k$ where its size is roughly 3^{-2k}.

With $k = 52500$ and $N = 70000$, I determined the Stokes constants (S_1, S_2) at **50000 digit** precision. This took about **3 minutes**. [*Here endeth the first lesson.*]

Non-linear recursion from a Dyson-Schwinger equation:
In **quantum field theory** we expand encounter **formal power series** whose coefficients are **integrals** whose integrands are specified by **Feynman diagrams**. Very often, the coefficients increase **factorially**. Here I shall deal with a case where the coefficients are rational numbers from which we obtain an **integer sequence**.

Symbolically, the series is generated by this picture



which generates T_n Feynman diagrams with n **loops**, where **A000081** at OEIS gives

$T_n =$ 1, 1, 2, 4, 9, 20, 48, 115, 286, 719, 1842, 4766, 12486, 32973, 87811, 235381 ...

These are the also number of **unlabelled rooted trees** with n nodes. Asymptotically,

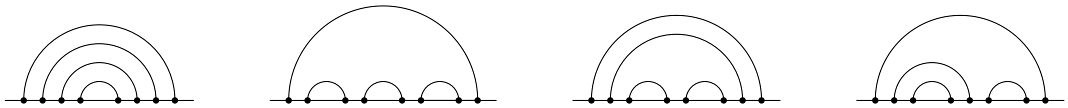
$$T_n = \frac{b}{n^{3/2}} c^n (1 + O(1/n))$$

$$b = 0.43992401257102530404090339143454476479808540794011\ldots$$

$$c = 2.95576528565199497471481752412319458837549230466359\ldots$$

which is quite benign. Yet the **contributions** from T_n diagrams grow **factorially**.

At 4 loops, we have a **rainbow**, a **chain** and two more **interesting** diagrams:



The sum of rainbows **converges**. Chains can be summed by **Borel transformation**.

$$\gamma_{\rm rainbow} = \frac{3-\sqrt{5+4\sqrt{1+a}}}{2} = -\frac{a}{6} + 11\,\frac{a^2}{6^3} - 206\,\frac{a^3}{6^5} + 4711\,\frac{a^4}{6^7} + O(a^5)$$

$$\gamma_{\rm chain} = -\int_0^\infty \frac{6\exp(-6z/a){\rm d}z}{(z+1)(z+2)(z+3)} = -\frac{a}{6} + 11\,\frac{a^2}{6^3} - 170\,\frac{a^3}{6^5} + 3450\,\frac{a^4}{6^7} + O(a^5)$$

$$\gamma \sim \sum_{n>0} G_n \frac{(-a)^n}{6^{2n-1}} = -\frac{a}{6} + 11\,\frac{a^2}{6^3} - 376\,\frac{a^3}{6^5} + \mathbf{20241}\,\frac{a^4}{6^7} + O(a^5)$$

with **large integers** G_n in the **alternating** asymptotic series for $\gamma(a)$. Note that $G_4 = 20241 > 4711 + 3450$, because of two further diagrams, above. The sequence `1, 11, 376, 20241, 1427156, 121639250...` is **A051862** at the OEIS.

Dirk **Kreimer** and I showed that these numbers are generated by a **third-order** differential equation with **quartic non-linearity**,

$$8a^3\gamma\left\{\gamma^2\gamma''' + 4\gamma\gamma'\gamma'' + (\gamma')^3\right\} + 4a^2\gamma\left\{2\gamma(\gamma-3)\gamma'' + (\gamma-6)(\gamma')^2\right\}$$
$$+ 2a\gamma(2\gamma^2 + 6\gamma + 11)\gamma' - \gamma(\gamma+1)(\gamma+2)(\gamma+3) = a.$$

Padé–Borel summation with alternating signs: We sought to resum the factorially divergent alternating series by an Ansatz

$$\gamma(a) = -\frac{a}{6\Gamma(\beta)}\int_0^\infty B(ax/3)\exp(-x)x^{\boldsymbol{\beta}-1}\mathrm{d}x, \quad B(z) = \frac{N(z)}{D(z)}.$$

The expansion coefficients of the **Borel transform** $B(z) = 1 + O(z)$ are obtained from those those of $\gamma(a)/a$ by dividing the latter by factorially increasing factors, producing a function expected to have a **finite** radius of convergence in the Borel variable z, with singularities on the **negative** z-axis, as for the sum of chains.

The **Padé** trick is to convert the expansion of B, up to n loops, into a **ratio** N/D of polynomials of degrees close to $n/2$. Then one can **check** how well this method reproduces G_{n+1}. We found that this works rather well with $\boldsymbol{\beta} \approx \mathbf{3}$. Gerald **Dunne** has recently shown that this method works even **better** with $\boldsymbol{\beta} = \mathbf{35/12}$, for reasons that I shall now **explain**.

Trans-series and resurgent hyperasymptotics:
Michael **Borinsky** and I considered the **sign-constant** asymptotic expansion

$$g_0(x) \sim \sum_{n\geq 0} A_n x^n = \frac{1}{2} + \frac{11}{24}x + \frac{47}{36}x^2 + \frac{2249}{384}x^3 + \frac{356789}{10368}x^4 + \frac{60819625}{248832}x^5 + O(x^6)$$

that formally solves the **non-linear** differential equation for $g(x) = -\gamma(-3x)/x$,

$$(g(x)P - 1)(g(x)P - 2)(g(x)P - 3)g(x) = -3, \quad P = x\left(2x\frac{\mathrm{d}}{\mathrm{d}x} + 1\right)$$

Ar large n, the expansion coefficients A_n behave as

$$A_n = S_1 \Gamma\left(n + \mathbf{\frac{35}{12}}\right)\left(1 - \mathbf{\frac{97}{48}}\left(\frac{1}{n}\right) + O\left(\frac{1}{n^2}\right)\right),$$

with a **Stokes constant** $S_1 = 0.087595552909179124483795447421262990627388\ldots$
which can be determined **empirically** by considering a solution

$$g(x) = g_0(x) + \sigma_1 x^{-\beta}\exp(-1/x)h_1(x) + O(\sigma_1^2)$$

and retaining terms **linear** in σ_1 in the nonlinear ODE.

This yields a **linear homogeneous** ODE for $h_1(x)$, which permits a solution that is **finite and regular** at $x = 0$ if and **only if** $\beta = \mathbf{\frac{35}{12}}$. Normalizing σ_1 by setting $h_1(0) = -1$, we obtain the expansion of

$$h_1(x) \sim \sum_{k\geq 0} B_k x^k = -1 + \mathbf{\frac{97}{48}}x + \frac{53917}{13824}x^2 + \frac{3026443}{221184}x^3 + \frac{32035763261}{382205952}x^4 + O(x^5)$$

which gives the **first-instanton** correction to the perturbative solution, suppressed by $\exp(-1/x)$. Developing the series A_n and B_k, I determined **3000 digits** of S_1 in

$$A_n \sim -S_1 \sum_{k\geq 0} \Gamma\left(n + \frac{35}{12} - k\right) B_k.$$

This is an example of **resurgence**: information about A_n resurges in B_k, and vice versa, because both $A(x) = g_0(x)$ and $B(x) = h_1(x)$ know about the **same** physics. **Hyperasymptotic** expansions involve the study of how B_n behaves at large n, which involves another set of numbers C_k, at small k, and so on, and so on.

Large A's need smaller B's, especially to guide them,
and larger B's need smaller C's, and so ad infinitum.

Trans-series: Hyperasymptotic investigation involves terms suppressed by $\exp(-m/x)$, with **action** $m > 1$. For this **third-order** ODE, there are **3 solutions** to the **linearized** problem, namely

$$g(x) = g_0(x) + \sigma_m \left(x^{-\frac{35}{12}} e^{-\frac{1}{x}}\right)^m h_m(x) + O(\sigma_m^2), \quad m \in \{1, 2, 3\},$$

with $h_2/x^5 = C$ and $h_3/x^5 = D$ finite and regular near the origin. Then we use linear ODEs to develop the expansions

$$C(x) = h_2(x)/x^5 = -1 + \frac{151}{24}x - \frac{63727}{3456}x^2 + \frac{7112963}{82944}x^3 - \frac{7975908763x}{23887872}x^4 + O(x^5),$$
$$D(x) = h_3(x)/x^5 = -1 + \frac{227}{48}x + \frac{1399}{4608}x^2 + \frac{814211}{73728}x^3 + \frac{3444654437}{42467328}x^4 + O(x^5).$$

But that is not the end of the story. We have solutions involving **products** of σ_m. We are developing a **trans-series**. Écalle tells use to expect a **triple** expansion in powers of x, $\exp(-1/x)$ and $\log(x)$. The coefficients come from the same ODE. They know about each other. Structure at $\exp(-m/x)$ will **resurge** at different actions.

The terms in the **trans-series** with action $m < 4$ are of the form

$$g = \sum_{m\geq 0} g_m \left(x^{-\frac{35}{12}}\, e^{-\frac{1}{x}}\right)^m, \quad L = \frac{21265}{2304} x^5 \log(x),$$
$$g_0 = A, \quad g_1 = \sigma_1 B, \quad g_2 = \sigma_2 x^5 C + \sigma_1^2 (F + CL),$$
$$g_3 = \sigma_3 x^5 D + \sigma_1\sigma_2 x^5 E + \sigma_1^3 (I + (D+E)L).$$

Denoting the coefficients of x^n in functions by subscripts, we found that

$$B_n \sim -2S_1 \sum_{k\geq 0} F_k \Gamma(n + \tfrac{35}{12} - k)$$
$$+ 4S_1 \sum_{k\geq 0} C_k \Gamma(n - \tfrac{25}{12} - k) \left(\tfrac{21265}{4608} \psi(n - \tfrac{25}{12} - k) + d_1\right),$$
$$d_1 = -43.33263472825075592450071739031938070346072802227 8\ldots$$

with $\psi(z) = \Gamma'(z)/\Gamma(z) = \log(z) + O(1/z)$, shows the $m = 1$ term, at large n, looking **forward** to $m = 2$ terms, at small k.

For the asymptotic expansion of the **second-instanton** coefficients, we found

$$C_n \sim -S_1 \sum_{k\geq 0} E_k \Gamma(n+\tfrac{35}{12}-k) + S_3 \sum_{k\geq 0} B_k (-1)^{n-k} \Gamma(n+\tfrac{25}{12}-k).$$

The first sum looks **forwards** to $m=3$ in the trans-series, where coefficients of

$$E(x) = -4 + \tfrac{371}{12}x - \tfrac{111785}{1152}x^2 + \tfrac{8206067}{18432}x^3 - \tfrac{18251431003}{10616832}x^4 + O(x^5)$$

appear. The second sum has **alternating** signs, looks **backwards** to $m=1$ and is **suppressed** by a factor of $1/n^{5/6}$. Likewise,

$$\begin{aligned} F_n \sim & -3S_1 \sum_{k\geq 0} I_k \Gamma(n+\tfrac{35}{12}-k) \\ & + 2S_1 \sum_{k\geq 0} (3D_k + 2E_k) \Gamma(n-\tfrac{25}{12}-k) \left(\tfrac{21265}{4608}\psi(n-\tfrac{25}{12}-k) + d_1\right) \\ & - 2S_3 \sum_{k\geq 0} B_k (-1)^{n-k} \Gamma(n-\tfrac{35}{12}-k) \left(\tfrac{21265}{4608}\psi(n-\tfrac{35}{12}-k) + f_1\right) \end{aligned}$$

looks forwards to I_k, D_k and E_k, at $m=3$, and backwards to B_k at, $m=1$. On the next slide, I exhibit the **whole story**, as compactly as possible.

$$g(x) = \sum_{m=0}^{\infty} \left(x^{-\frac{35}{12}} \, e^{-\frac{1}{x}} \right)^m \sum_{i=0}^{\lfloor m/2 \rfloor} \sum_{j=0}^{\lfloor (m-2i)/3 \rfloor} \sigma_1^{m-2i-3j} \widehat{\sigma}_2^i \widehat{\sigma}_3^j x^{5(i+j)} \sum_{n \geq 0} a_{i,j}^{(m)}(n) x^n,$$

$$\widehat{\sigma}_2 = \sigma_2 + \tfrac{21265}{2304} \sigma_1^2 \log(x), \quad \widehat{\sigma}_3 = \sigma_3 + \tfrac{21265}{2304} \sigma_1^3 \log(x),$$

$$a_{i,j}^{(m)}(n) \sim -(s+1) S_1 \sum_{k \geq 0} a_{i,j}^{(m+1)}(k) \Gamma(n + \tfrac{35}{12} - k)$$

$$+ S_1 \sum_{k \geq 0} \left(4(i+1) a_{i+1,j}^{(m+1)}(k) + 6(j+1) a_{i,j+1}^{(m+1)}(k) \right) \Gamma(n - \tfrac{25}{12} - k) \left(\tfrac{21265}{4608} \psi(n - \tfrac{25}{12} - k) + d_1 \right)$$

$$+ \tfrac{1}{4} S_3 \sum_{k \geq 0} \left(4(s+1) a_{i-1,j}^{(m-1)}(k) + 6(j+1) a_{i-2,j+1}^{(m-1)}(k) \right) (-1)^{n-k} \Gamma(n + \tfrac{25}{12} - k)$$

$$- 2(s - 2i - 1) S_3 \sum_{k \geq 0} a_{i,j}^{(m-1)}(k) (-1)^{n-k} \Gamma(n - \tfrac{35}{12} - k) \left(\tfrac{21265}{4608} \psi(n - \tfrac{35}{12} - k) + f_1 \right)$$

$$- S_3 \sum_{k \geq 0} \left(8(i+1) a_{i+1,j}^{(m-1)}(k) + 6(j+1) a_{i,j+1}^{(m-1)}(k) \right) (-1)^{n-k} \Gamma(n - \tfrac{95}{12} - k) Q(n - \tfrac{95}{12} - k)$$

$$- (f_1 - c_1) S_3 \sum_{k \geq 0} \left(2(i+1) a_{i+1,j-1}^{(m-1)}(k) + 6(i+j) a_{i,j}^{(m-1)}(k) \right) (-1)^{n-k} \Gamma(n - \tfrac{35}{12} - k),$$

$$s = m - 2i - 3j, \quad Q(z) = \left(\tfrac{21265}{4608} \right)^2 \left(\psi^2(z) + \psi'(z) \right) + 2c_1 \left(\tfrac{21265}{4608} \right) \psi(z) + c_2.$$

Comments and conclusions:

1. The **linear** problem from **game theory** has simple hyperasymptotics. Its integer sequence has a pair of **conjugate** governors, which also govern **each other**.

2. The integer sequence from **quantum field theory** comes from a **non-linear** recursion. Its **trans-series** exhibits **17 types of resurgence**, intensively tested at **high precision**, for all **actions** $m \leq 8$.

3. The **6 Stokes constants** have been determined to better than **1000 digits**.

4. Excellent **freeware**, from `Pari-GP` in Bordeaux, was vital to this enterprise.

5. The presence of **logarithms** in the trans-series may be ascribed to **resonance** between three **equally spaced** instantons.

6. I have been guided by advice from **Gerald Dunne** and encouraged by programmes and workshops on *Applicable Resurgent Asymptotics* at the **Isaac Newton Institute**, in Cambridge, and on *Resurgence and Modularity in QFT and String Theory* at the **Galileo Galilei Institute**, in Florence.