Generating Functions for Enumerating Spanning Trees

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(joint work with Doron Zeilberger)

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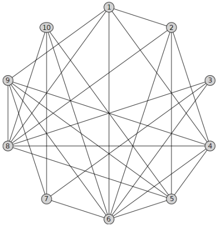
Plan of attack

- Find a nice (ordered) graph family
- Count the number of spanning trees for a lot of members (or count total number of leaves) to generate sequence terms
- Guess the Rational Generating Function now that you have sufficiently many terms
- ???
- Profit.

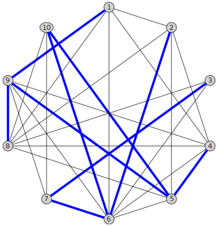
What is a Graph?

A **graph** G is a pair (V, E) of vertices and edges between them.
We write $V(G)$ for the vertices of G and $E(G)$ for its edges.



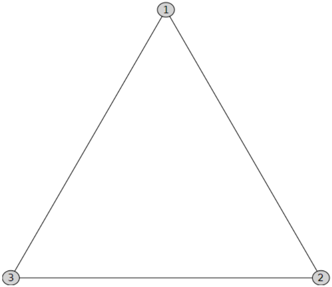
What is a Spanning Tree?

A **spanning tree** T of G is a tree (connected and no cycles) such that $V(T) = V(G)$.



How do we count spanning trees?

In our project, we counted *labelled* spanning trees. For example, K_3 would have 3 trees, not 1.



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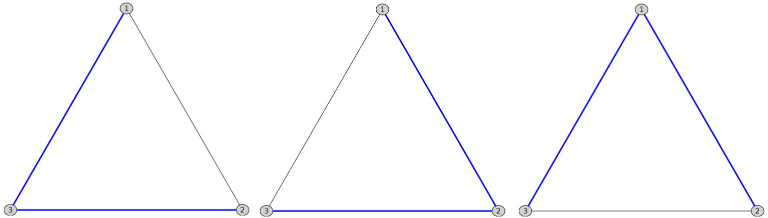

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$$\begin{pmatrix} \deg(v_1) & -\mathbf{1}_{(v_1 \sim v_2)} & \dots & -\mathbf{1}_{(v_1 \sim v_n)} \\ -\mathbf{1}_{(v_1 \sim v_2)} & \deg(v_2) & \dots & -\mathbf{1}_{(v_2 \sim v_n)} \\ \vdots & \vdots & \ddots & \vdots \\ -\mathbf{1}_{(v_1 \sim v_n)} & -\mathbf{1}_{(v_2 \sim v_n)} & \dots & \deg(v_n) \end{pmatrix}.$$

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Delete the row and a column of an entry. Take the determinant. Now you have computed the **number of spanning trees**, $\tau(G)$, of G.

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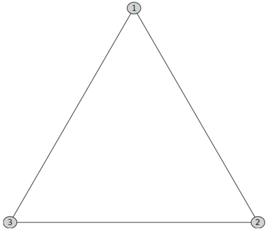
$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$



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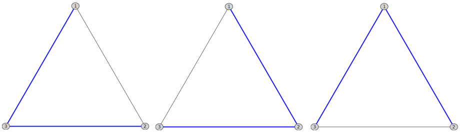
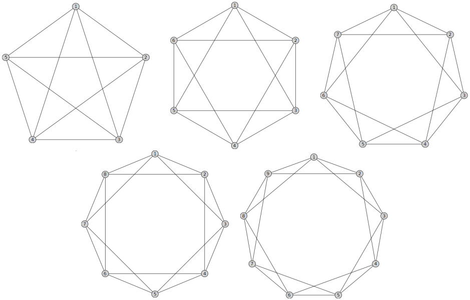
$$\left| \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \right| = 2^2 - 1 = 3$$



Graph families we considered

Powers of Cycles: Fix r. Then,

$$\mathcal{G}_r := \{ {C_n}^r \ : \ n \geq 2r+1 \}.$$

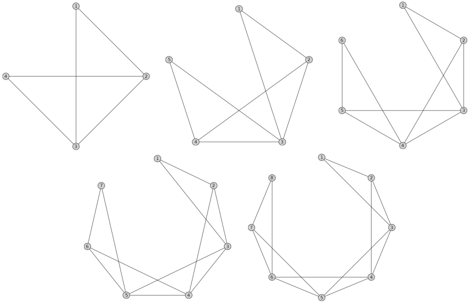


$r = 2$

Graph families we considered

Powers of Paths: Fix r. Then,

$$\mathcal{H}_r := \{{P_n}^r \;:\; n \geq r+2\}$$



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$$a_{n+r} = c_{r-1} a_{n+r-1} + \ldots + c_0 a_n$$

for all n, then we call it **C-finite** (of order r if $c_0 \neq 0$).

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Example: The Fibonacci Sequence $(1, 1, 2, 3, 5, 8, 13, \ldots)$ satisfies the relation $F_{n+2} = F_{n+1} + F_n$, so it is C-finite of order 2.

Why are you talking about C-finite sequences?

A sequence is C-finite satisfying the relation $a_{n+r} = c_{r-1}a_{n+r-1} + \ldots + c_0 a_n$ iff:

$$\sum a_n x^n = \frac{p(x)}{1 - c_{r-1}x + \ldots - c_1 x^{r-1} - c_0 x^r}.$$

($p(x)$ has degree $\leq r-1$ and depends on the initial terms of the sequence)

Example: The Fibonacci Sequence has relation $F_{n+2} = F_{n+1} + F_n$ and its generating function is

$$\sum F_n x^n = \frac{1}{1 - x - x^2}$$

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Some New Sequences

Number of spanning trees for $\mathcal{G}_r$ $(r \geq 4)$ and $\mathcal{H}_r$ $(r \geq 3)$ are not in the OEIS.

```
> H3:=SeqHnr(3,20);
  H4:=SeqHnr(4,20);
  H5:=SeqHnr(5,20);
```

H3 ≔ [75, 336, 1488, 6580, 29085, 128544, 568101, 2510716, 11096064, 49038840, 216726195, 957817168, 4233054171, 18707899800, 82679195856, 365399082748, 1614874071885, 7136904253920, 31541408222709, 139396634349556]

H4 ≔ [864, 5635, 35840, 226080, 1424736, 8975232, 56531412, 356045600, 2242419040, 14122994787, 88948032416, 560203336285, 3528214538112, 22221034368624, 139950209558628, 881419864147200, 5551266971808376, 34962412626016064, 220196633083726032, 1386819546435968365]

H5 ≔ [12005, 104448, 878688, 7272720, 59829840, 491863680, 4042376800, 33217265664, 272934155637, 2242522832400, 18425237837125, 151386977585232, 1243837315587760, 10219707278640384, 83967892455015936, 689902984258166320, 5668430017511142200, 46573358017220331264, 382659336814688436965, 3144032857765675104768]

```
> G4:=SeqHnr(4,20);
  G5:=SeqHnr(5,20);
  G6:=SeqHnr(6,20);
```

G4 ≔ [864, 5635, 35840, 226080, 1424736, 8975232, 56531412, 356045600, 2242419040, 14122994787, 88948032416, 560203336285, 3528214538112, 22221034368624, 139950209558628, 881419864147200, 5551266971808376, 34962412626016064, 220196633083726032, 1386819546435968365]

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G6 ≔ [196608, 2158569, 22806000, 235669280, 2407426560, 24478578432, 248773434624, 2527609743360, 25677708264000, 260836913033840, 2649495039624576, 26912151112299600, 273358076605088256, 2776609368905663295, 28203134687833190400, 286470442090292386905, 2909793491158628978928, 29555919864339961430800, 300211122260663167530240, 3049362618513184407642624]

Some Generating Functions

The generating function $f(t)$ *for the number of spanning trees in* $\mathcal{H}_3$ *is*

$$\frac{-16t^4+77t^3-33t^2+39t-75}{(t-1)(t^4-4t^3-t^2-4t+1)}.$$

The generating function $f(t)$ *for the number of spanning trees in* $\mathcal{G}_3$ *is*

$$\frac{N_3}{(t-1)^2(t^4+3t^3+6t^2+3t+1)^2(t^4-4t^3-t^2-4t+1)^2}$$

The generating function $f(t)$ *for the number of spanning trees in* $\mathcal{H}_4$ *is*

$$\frac{M_4}{(t^6-3t^5+6t^4-10t^3+6t^2-3t+1)(t^8-4t^7-17t^6+8t^5+49t^4+8t^3-17t^2-4t+1)}.$$

$$D_4=(t+1)^2(t^6-3t^5+6t^4-10t^3+6t^2-3t+1)^2(t^8-4t^7-17t^6+8t^5+49t^4+8t^3-17t^2-4t+1)^2$$
$$(t^{12}+3t^{11}+12t^{10}+28t^9-27t^8+36t^7-81t^6+36t^5-27t^4+28t^3+12t^2+3t+1)^2.$$

Counting Total Number of Leaves

How do we count the total number of leaves across all spanning trees of a graph? Delete a vertex v and count the spanning trees of $G - v$.

$$\sum_{T \in \mathcal{T}(G)} |\mathcal{L}(T)| = \sum_{v \in V(G)} \deg_G(v) \cdot \tau(G - v).$$

$\mathcal{L}(T)$ is the set of leaves a tree T.
$\mathcal{T}(G)$ is the collection of (labelled) spanning trees of G.
$\tau(G)$ is the number of spanning trees of G.

Counting Total Number of Leaves in Vertex-Transitive Graphs

A graph G is **vertex-transitive** if $G - u$ is isomorphic to $G - v$ for all $u, v \in V(G)$.

$$\sum_{T \in \mathcal{T}(G)} |\mathcal{L}(T)| = n \cdot \deg_G(v) \cdot \tau(G - v)$$

The talk is over

Thanks!