Powers of 2 in High-Dimensional Lattice Walks

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$$\cdots \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad \cdots$$

Begin at 0
make $2n$ steps
return to 0

How many ways?

$$W_1(n) = \binom{2n}{n}$$

What about 2D?
Begin at 0
make $2n$ steps
return to 0

How many ways?
We go left, right, up, down

$$W_2(n) = \sum_{x_1+x_2=n} \binom{2n}{x_1, x_1, x_2, x_2}$$

Choose positive/negative first
then directions

$$W_2(n) = \binom{2n}{n} \sum_{x_1+x_2=n} \binom{n}{x_1, x_2}^2$$

Vandermonde's identity

$$W_2(n) = \binom{2n}{n}^2$$

What about any number of dimensions?

$$W_d(n) = \sum_{x_1+\cdots+x_d=n} \binom{2n}{x_1, x_1, \ldots, x_d, x_d}$$

Choose positive/negative first
then directions

$$W_d(n) = \binom{2n}{n} \sum_{x_1+\cdots+x_d=n} \binom{n}{x_1, \ldots, x_d}^2$$

How many 2's are there in $W_d(n)$?

Kummer's theorem

How many 2's are there in $\binom{a+b}{a,b}$?

Look at a, b in binary

$$W_1(n) = \binom{2n}{n}$$

Let s be the number of 1's
in the binary expansion of n
The answer is s

What about 2D?

$$W_2(n) = \binom{2n}{n}^2$$

The answer is $2s$

What about 3D?

$$W_3(n) = \binom{2n}{n} \sum_{x_1+x_2+x_3=n} \binom{n}{x_1, x_2, x_3}^2$$

Apply Kummer to each summand

Count odds

Generalisation

For all odd d the answer is s

What about even $d = 2e$?
Fold by Vandermonde

$$W_d(n) = \binom{2n}{n} \text{ times}$$

$$\sum_{y_1+\cdots+y_e=n} \binom{2y_1}{y_1} \cdots \binom{2y_e}{y_e} \binom{n}{y_1, \ldots, y_e}^2$$

Same method works when e odd

For all $d \equiv 2 \pmod 4$ the answer is $2s$

What we know so far

$d \equiv 1 \pmod 2$	s
$d \equiv 2 \pmod 4$	$2s$
$d \equiv 4 \pmod 8$	?

Let's run some experiments

n	1	2	3	7	9	21	25
s	1	1	2	3	2	3	3
exp	3	3	10	12	6	9	11

What we know so far

$d \equiv 1 \pmod 2$	s
$d \equiv 2 \pmod 4$	$2s$
$d \equiv 4 \pmod 8$	$\geq 3s$
$d \equiv 8 \pmod{16}$	?

The truth

$d \equiv 1 \pmod{2}$	s
$d \equiv 2 \pmod{4}$	$2s$
$d \equiv 4 \pmod{8}$	$\geq 3s$
$d \equiv 8 \pmod{16}$	$\geq 3s+1$
$d \equiv 16 \pmod{32}$	$\geq 3s+2$

$$\cdots$$

We know the cases of equality, too

Focus on $d = 4$

The method generalises

$$W_4(n) = \binom{2n}{n} \text{ times}$$

$$\sum_{x+y=n} \binom{2x}{x}\binom{2y}{y}\binom{n}{x,y}^2$$

Consider just

$$\sum_{x+y=n} \binom{2x}{x}\binom{2y}{y}\binom{n}{x,y}^2$$

We want exponent $2s$

Apply Kummer to each summand
Some summands fine
Others not so much

Worst summands

$x + y = n$ without carries

Exponent is just s

We want exponent $2s$

What to do?

Consider worst ones first
Gather all together

$$\sum_{\substack{x+y=n \\ \text{carry-free}}} \binom{2x}{x}\binom{2y}{y}\binom{n}{x,y}^2$$

Surprise

Sum of worst ones has exponent $2s$

The method generalises
Gather all summands into groups
Each group has exponent $\geq 2s$
Focus on the worst group

Each summand has exponent s
We gather 2^s summands
Suddenly, exponent $2s$
We get a boost

Why so special?

$$\binom{2x}{x}\binom{2y}{y}\binom{n}{x,y}^2$$

Not that special

Let's run some experiments

When do we get a boost?

$$F(x, y) = \binom{n}{x, y}$$

$$F(x, y) = \binom{n}{x, y}^{a}$$

$$F(x, y) = \begin{pmatrix} 2x \\ x \end{pmatrix} \begin{pmatrix} 2y \\ y \end{pmatrix}$$

$$F(x, y) = \left[\begin{pmatrix} 2x \\ x \end{pmatrix} \begin{pmatrix} 2y \\ y \end{pmatrix} \right]^a$$

Products of such

$$F(x,y) = \left[\binom{2x}{x}\binom{2y}{y}\right]^a \binom{n}{x,y}^b$$

$$F(x, y) = \begin{pmatrix} 4x \\ 2x \end{pmatrix} \begin{pmatrix} 4y \\ 2y \end{pmatrix}$$

$$F(x, y) = \left[\begin{pmatrix} 4x \\ 2x \end{pmatrix} \begin{pmatrix} 4y \\ 2y \end{pmatrix} \right]^a$$

$$\cdots$$

Conjecture

We get a boost for all such $F(x, y)$

Proof?

First we tidy up a bit

What is $F(x, y)$, really?

The right building blocks are the factorials

$$F(x, y) = \frac{(2x)!\ (2y)!\ n!^2}{x!^4\ y!^4}$$

Take out common factors

$$f_i(x, y) = \text{odd part of } (2^i x)!\ (2^i y)!$$

$$F(x, y) = f_0(x, y)^{-4}\ f_1(x, y)$$

Division doesn't matter
Focus on products of f_i's
raised to positive powers

Induction

One more tweak

Focus on products of f_i's

raised to positive powers

times polynomial $P(x, y)$

Example

$$F(x, y) = f_0(x, y)$$

First case: Even $n = 2m$

Carry-free $x + y = n$ becomes

x		0
$+\, y$		0
$=\, n$		0

$$\sum_{\text{carry-free } n} f_0(x, y) =$$

$$\sum_{\text{carry-free } m} f_0(2x, 2y) =$$

$$\sum_{\text{carry-free } m} f_1(x, y)$$

Second case: Odd $n = 2m + 1$

Carry-free $x + y = n$ becomes

x		$0^{(1)}$
$+\, y$		$1^{(0)}$
$=\, n$		1

$$\sum_{\text{carry-free } n} f_0(x, y) =$$

$$\sum_{\text{carry-free } m} \left[f_0(2x+1, 2y) + f_0(2x, 2y+1) \right] =$$

$$\sum_{\text{carry-free } m} \Big[(2x+1) f_0(2x, 2y) +$$

$$+ (2y+1) f_0(2x, 2y)\Big] =$$

$$\sum_{\text{carry-free } m} (2x+2y+2) f_1(x, y) =$$

$$2 \cdot \sum_{\text{carry-free } m} (x+y+1) f_1(x,y)$$

The induction step is complete

Wrapping up

We can gather the worst into a group

We can gather all summands into groups

Divisibility follows

The method generalises

$$F(z_1, \ldots, z_k)$$

Divisibility follows for all d

Further tinkering gives us the cases of equality

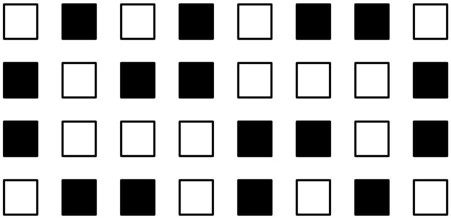
Fun by-products

The sum of all odd binomial coefficients in row n is divisible by 2^s

Fun by-products
Sums of powers
Sums of multinomial coefficients
Other primes

Why do we care about
the number of 2's in $W_d(n)$?

Bhattacharya's conjecture



Let $B(m, n)$ be the number of ways for size $2m \times 2n$

How many 2's are there in $B(m, n)$?

The conjecture is $s_2(m)s_2(n)$

Connection to lattice walks

$$B(1, n) = W_1(n)$$
$$B(2, n) = W_3(n)$$

Confirmed for

$m = 2^k$, any n: Easy

$m = 2^k + 1$, any n: Not so easy

Thank you