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A TREATISE

ON THE

CALCULUS OF FINITE DIFFERENCES.

BY

GEORGE BOOLE, D.C.L.

HONORARY MEMBER OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY;
PROFESSOR OF MATHEMATICS IN THE QUEEN'S UNIVERSITY, IRELAND.

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1860.

Thus we can integrate any rational fraction of the form

$$\frac{\phi(x)}{u_x u_{x+1} \ldots u_{x+m}},$$

u_x being of the form $ax+b$, and $\phi(x)$ a rational and integral function of x of a degree lower by at least two unities than the degree of the denominator. For, expressing $\phi(x)$ in is obvious.

4th Form. Functions of the form $a^x \phi(x)$ in which $\phi(x)$ is rational and integral.

4—2

Since $\Delta a^x = (a-1) a^x$, we have

$$\Sigma a^x = \frac{a^x}{a-1} + C.$$

To deduce $\Sigma a^x \phi(x)$ we may now employ either a method of integration by parts or a symbolical method founded upon the relations between the exponential a^x and the symbol Δ.

To integrate by parts we have,

$$\text{since} \quad \Delta u_x v_x = u_x \Delta v_x + v_{x+1} \Delta u_x,$$

$$u_x \Delta v_x = \Delta u_x v_x - v_{x+1} \Delta u_x,$$

therefore

$$\Sigma u_x \Delta v_x = u_x v_x - \Sigma v_{x+1} \Delta u_x \quad \text{.................. (7)},$$

the theorem required. Applying this we have

$$\Sigma \phi(x)\, a^x = \phi(x) \frac{a^x}{a-1} - \Sigma \frac{a^{x+1}}{a-1} \Delta \phi(x)$$

$$= \frac{1}{a-1} \left\{ \phi(x)\, a^x - a \Sigma a^x \Delta \phi(x) \right\}.$$

Thus the integration of $a^x \phi(x)$ is made to depend upon that of $a^x \Delta \phi(x)$; this again will by the same method depend upon that of $a^x \Delta^2 \phi(x)$, and so on. Hence $\phi(x)$ being by hypothesis rational and integral, the process may be continued until the function under the sign Σ vanishes. This will happen after $n+1$ operations if $\phi(x)$ be of the n^{th} degree; and the integral will be obtained in finite terms.

But the symbolical method above referred to leads to the same result by a single operation.

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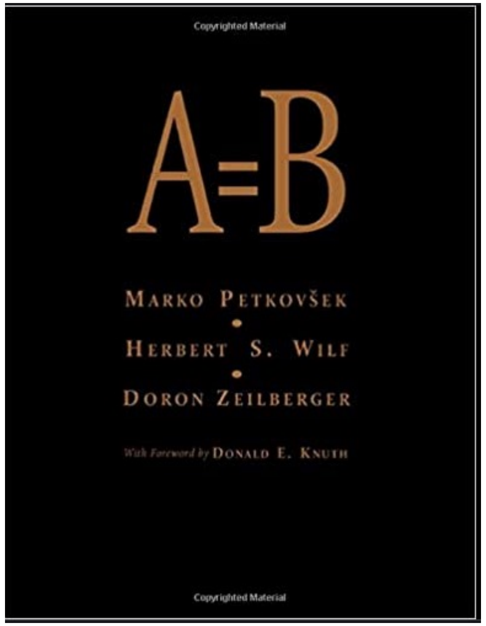

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Indefinite Hypergeometric Sums in MACSYMA*

R. Wm. Gosper, Jr.
XEROX Palo Alto Research Center

ABSTRACT

We present a MACSYMA function which, given the summand

(A) $$a_n = \Delta g(n) = g(n+1) - g(n)\,,$$

finds $g(n)$, the "indefinite sum", within an additive constant, provided that $g(n+1)/g(n)$ is a rational function of n. We then have the identity

(B) $$\sum_{n=p}^{q} a_n = g(q+1) - g(p)\,.$$

The Algorithm

The only significant problem is to solve the rational functional equation

(func) $$\frac{a_{n+1}}{a_n} = \frac{f(n)+1}{f(n+1)}$$

for f. Since this followed from differencing $g(n) = f(n)\,a_n = f(n)\triangle g(n)$, we have

$$f(n) = \frac{g(n)}{g(n+1)-g(n)} = \frac{1}{\frac{g(n+1)}{g(n)} - 1}$$

which is rational when $g(n+1)/g(n)$ is. Because we have no boundary condition to satisfy, equation (func) is easier to satisfy than a first order linear recurrence with polynomial coefficients. In fact, if $f(n)$ is a solution, If f is a rational function, then the quotients from Euclid's algorithm (using polynomial division) form the terms of its continued fraction:

$$f(n) = p_1(n) + \cfrac{1}{p_2(n) + \cfrac{1}{p_3(n) + \cfrac{1}{\ddots \; p_k(n)}}}.$$

Our MACSYMA algorithm successively determines $p_1, p_2, \ldots$, with the proviso that no p_i be constant for $i > 1$, so as to guarantee the uniqueness of the representation.

As a result, I patched the algorithm to only determine q of its $q+1$ undetermined coefficients on non terminal terms where $q > 1$, thus treating all such cases in the manner of (weirdo). This seemed to repair the problem, at the cost of exhausting list storage capacity on certain cases that had formerly worked. Fortunately, on 20 April 1977, all of this kludgery was rendered obsolete when I found a decision procedure for this problem. (A discrete analog to the Risch algorithm for indefinite integration.) The procedure is

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Mathematics

Decision procedure for indefinite hypergeometric summation

(algorithm/binomial coefficient identities/closed form/symbolic computation/linear recurrences)

R. WILLIAM GOSPER, JR.

Xerox Palo Alto Research Center, Palo Alto, California 94304

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ABSTRACT Given a summand a_n, we seek the "indefinite sum" $S(n)$ determined (within an additive constant) by

$$\sum_{n=1}^{m} a_n = S(m) - S(0) \qquad [0]$$

or, equivalently, by

$$a_n = S(n) - S(n-1). \qquad [1]$$

An algorithm is exhibited which, given a_n, finds those $S(n)$ with the property

$$\frac{S(n)}{S(n-1)} = \text{a rational function of } n. \qquad [2]$$

Method

If $S(n)/S(n-1)$ is a rational function of n, then by Eq. **1** the term ratio

$$\frac{a_n}{a_{n-1}} = \frac{S(n) - S(n-1)}{S(n-1) - S(n-2)} = \frac{\dfrac{S(n)}{S(n-1)} - 1}{1 - \dfrac{S(n-2)}{S(n-1)}} \qquad [4]$$

must also be a rational function of n. (We exclude the degenerate case where a_n is identically zero.) Express this ratio as

$$\frac{a_n}{a_{n-1}} = \frac{p_n}{p_{n-1}} \frac{q_n}{r_n}, \qquad [5]$$

where p_n, q_n, and r_n are polynomials in n subject to the following condition:

$$\gcd(q_n, r_{n+j}) = 1, \qquad [6]$$

for all non-negative integers j.

It is always possible to put a rational function in this form, for if $\gcd(q_n, r_{n+j}) = g(n)$, then this common factor can be eliminated with the change of variables

$$q_n' \leftarrow \frac{q_n}{g(n)}, \; r_n' \leftarrow \frac{r_n}{g(n-j)},$$
$$p_n' \leftarrow p_n g(n) g(n-1) \ldots g(n-j+1), \qquad [6']$$

which leaves the term ratio unchanged. The values of j for which such gs exist can be readily detected as the non-negative integer roots of the resultant of q_n and r_{n+j} with respect to n.

We now write

$$S(n) = \frac{q_{n+1}}{p_n} f(n) a_n, \qquad [7]$$

where $f(n)$ is to be determined. By using Eq. **1**,

$$f(n) = \frac{p_n}{q_{n+1}} \frac{S(n)}{S(n) - S(n-1)} = \frac{p_n}{q_{n+1}} \frac{1}{1 - \dfrac{S(n-1)}{S(n)}},$$

so $f(n)$ is a rational function of n whenever $S(n)/S(n-1)$ is. By substituting Eq. **7** into Eq. **1**, we get

$$a_n = \frac{q_{n+1}}{p_n} f(n) a_n - \frac{q_n}{p_{n-1}} f(n-1) a_{n-1}.$$

Multiplying this through by p_n/a_n, and using Eq. **5**, we have

$$p_n = q_{n+1} f(n) - r_n f(n-1), \qquad [8]$$

the functional equation for f.

THEOREM. *If* $S(n)/S(n-1)$ *is a rational function of* n, *then* $f(n)$ *is a polynomial.*

At the same time ...

Moenck, R.: On computing closed forms for summations. In: Proceedings of the 1977 MACSYMA Users' Conference, pp. 225–236 (1977)

Remembering from section 3 that powers are not nice forms for summation, we define a factorial operator on a function:

(9) $$[f(x)]_k = f(x) \cdot f(x-1) \cdot f(x-2) \ldots f(x-k+1) \quad \text{for } k>0 \,.$$

We can extend this operator by noticing:

(10) $$[f(x)]_k = [f(x)]_\ell \cdot [f(x-\ell)]_{k-\ell}$$

If we define $[f(x)]_0 = 1$ and assert that (10) is an identity then substituting $k=0$ we get:

(11) $$[f(x)]_{-\ell} = \frac{1}{[f(x+\ell)]_\ell}$$

We will call the value of k or ℓ in equations 9 and 11, the factorial degree of function, because of its parallel to the "power" degree. We now proceed to examine the differences of factorials.

(12) $$\Delta[f(x)]_k = [f(x)]_{k-1} \underset{k}{\overset{x}{\Delta}} f(x-k+1) \;,\; k>0 \,.$$

$$Y(x+1)r(x) - Y(x) = 1, \tag{3}$$

$$zA(x)y(x+1) - B(x-1)y(x) = C(x). \tag{4}$$

$$G(x) = F(x)\frac{B(x-1)y(x)}{C(x)}. \tag{5}$$

Then substitution $y(x) = \tilde{y}(x)[p(x)]_{k+1}$ into (4) gives new equation

$$zA(x)p(x+1)\tilde{y}(x+1) - B(x-1)p(x-k)\tilde{y}(x) = \tilde{C}(x) \tag{6}$$

which has polynomial solution $\tilde{y}(x)$ and $y(x)$ in (5) can be replaced by $\tilde{y}(x)$, $C(x)$ in (5) can be replaced by $\tilde{C}(x)$ (effectively realizing cancellation of unnecessary common factor in the numerator and denominator of (5)). If the degree bound for $y(x)$ in (4) is N, then the degree bound for $\tilde{y}(x)$ in (6) is $N - (k+1)\deg p(x)$.

Take as an example of extreme case the following summation problem

$$\sum_x 2^x \frac{ax - x - 1000}{(x + 1000)\,x}.$$

Dispersion $\rho = 1000$ in this case, and the operator $M(E)$ in (24) is $\frac{a}{2^{1000}} E^{1000} - 1$. The quotient will have 1000 nonzero terms, and the remainder will be equal to $\frac{a}{2^{1000}} - 1$. In order for the problem to be summable a has to be equal to 2^{1000}. In this case all coefficients of the quotient will be equal to 1, the numerator of the input will have to have the size exponential in the size of the denominator, and expanded form of the result will have size polynomial in the size of the input.