Locality preserving hash functions, a partial order and tiles in binary space

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Finding two needles in a haystack

The problem

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Plant a pair $(x^{(i)}, x^{(j)})$ where, Hamming distance (number of bits that disagree) $d_H(x^{(i)}, x^{(j)}) = k \ll n$, and everything else is random.
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- Find $x^{(i)}, x^{(j)}$. 
Have we gone down a “Rabbit Hole”?

- The isoperimetric inequality for the Hamming Cube.
- Syndrome Decoding.
- An interesting partial order.
- Discrete tiles in a binary space.
- Fast Hadamard Transform.
- Linear Programming.
- Bin Packing.
A first attempt

Comments

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- For $N = 10^9$ that’s a lot of work.
- Can we do better?
A better (?) idea

- Use a “hash” function \( f : \mathbb{B}^n \rightarrow \mathbb{B}^r \).
- Put string \( x \) into the “bucket” labeled with \( f(x) \).
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Channels and Codes

- The *Binary Symmetric Channel* BSC($p$) takes each bit and flips it independently with probability $p$.

- Error detecting: if $x \in S$ and $\tilde{x} \notin S$ we've detected an error.

- Error correcting: $x \in S$, find $\hat{x}$ "closest" to $\tilde{x}$. 

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Probability of disagreement

\[ \mathcal{F}_S(p) : \text{probability if } x \in S \text{ is random that } \tilde{x} \in S. \]

\[ F_S(t) := \sum_{i=0}^{n} A_i(S)t^i, \quad A_i(S) := \# \{ x, y \in S : d_H(x, y) = i \} \]

\[ \mathcal{F}_S(p) := \frac{1}{|S|} (1 - p)^n F_S(p/(1 - p)) \]
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- **Goal**: Find \( S \subset \mathbb{B}^n, |S| = 2^{n-r} \) which maximizes \( \mathcal{F}_S(p) \).
Equivalence

- \( \sigma \in \mathcal{S}_n \): a permutation.
- \( \sigma(x) \) permutes the coordinates of \( x \).
- Note: \( F_{\sigma(S) \oplus a}(p) = F_S(p) \), where \( a \in \mathbb{B}^n \), \( \oplus \) is mod 2 addition of coordinates.
- We will say that \( S \) and \( \sigma(S) \oplus a \) are isomorphic.
- Thus \( P(f_{\sigma,a}) = P(f) \) where \( f_{\sigma,a}(x) = f(\sigma(x) \oplus a) \).
- Note: If \( S \) is “good” we can define a hash function \( f \) from it if it’s a tile: \( \mathbb{B}^n \) is a disjoint union of translates of \( S \) using \( \oplus \).
- Index translates by elements of \( \mathbb{B}^r \), map \( x \) to index of translate containing it.
The question I was asked

Projection: \( \pi : \mathbb{B}^n \to \mathbb{B}^r \) be \( \pi((x_1, \ldots, x_n)) = (x_1, \ldots, x_r) \).

Question: Can we do better than using \( \pi \)?

Answer: It depends on \( p \).
The isoperimetric theorem for the Hamming Cube

**Theorem (Isoperimetric Theorem (Harper))**

If $S \subset \mathbb{B}^n$, let $e(S) = \#\{x \in S, y \not\in S : d_H(x, y) = 1\}$. Then

$$e(S) \geq \frac{1}{2} |S| \log_2 |S|,$$

with equality if and only if $S$ is isomorphic to $(\ast, \ldots, \ast, 0, \ldots, 0)$, a subcube.
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Proof.
Use the isoperimetric inequality for the Hamming cube. Note that \( A_0(S) = |S|, A_1(S) = n|S| - e(S) \).
Doing better than projection

- \( C \subset \mathbb{B}^n \): a linear subspace of dimension \( r \).
Doing better than projection

- $C \subset \mathbb{B}^n$: a *linear* subspace of dimension $r$.
- *Check matrix*: $A$: $x \in C \iff Ax = 0$. 

Syndrome: Given $\tilde{x}$ calculate $A\tilde{x}$. Gives the coset of $C$ containing $\tilde{x}$. 

- For each coset $a \oplus C$ give $y \in a \oplus C$ of minimum Hamming weight: Coset Leader.
- Use the set of coset leaders as a region $S$.
- Theorem: Asymptotically this beats projection for a random code of fixed rate.

For the Golay code $G_{F_{\text{G}}}(t) := 2048 + 11684t + 128524t^2 + 226688t^3$, Better than projection when $p \geq 0.2555$. 

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For the Golay code $\mathcal{G}$

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Optimal Regions

Definition (Optimal Region)

Let $S \subset \mathbb{B}^n$. Say that $S$ is optimal at $t \in (0, 1)$ if $F_S(t) \geq F_{S'}(t)$ for all $S' \subset \mathbb{B}^n, |S'| = |S|$. 

$S$ is optimal if it is optimal at some $t \in (0, 1)$.

Theorem (Optimal Region Theorem (Gordon, Miller, Ostapenko))

An optimal subset $S \subset \mathbb{B}^n$ is isomorphic to an order ideal in the partial order $\leq_R$ (defined below).

Proof.

Uses the "shifting" and "compression" functions of Erdős-Ko-Rado from extremal set theory. Looks at local failures to be an order ideal, and corrects them.
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- Transitive: \( x \preceq y, y \preceq z \Rightarrow x \preceq z \).

Note: Not every pair \( x, y \in S \) may be comparable.
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- Define: \( x \preceq_R y \) if

\[
I(x)_1 \leq I(y)_1, \ldots, I(x)_k \leq I(y)_k,
\]

where \( k = \min(|I(x)|, |I(y)|) \).
What’s in a name?

The partial order $\preceq_R$ has many names.

- Kündgen: *right-shifted partial order*
- Stanley, Proctor (and others): $M(n)$ (the poset name).
- Ahlswede, Tamm: pushing order.

Has many interesting connections: partitions, Coxeter Groups.
Order Ideals

- **Order Ideal:** A subset $T \subset S$ where $x \in S$, $y \preceq x \Rightarrow y \in S$.
- **Generators:** $T \subset S$. $\langle T \rangle := \{ x \in S : \exists y \in T, x \preceq y \}$.
- **Principal ideal:** $\langle \{x\} \rangle$: one generator.
Finding all order ideals of a given size

- Squire: a recursion to find all order ideals of a poset.
- Number of all ideals grows too quickly, but we’re only interested those of limited size.
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Principal ideals of size \( n \) in \( M(n) \)

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Order ideals of size $n$ in $M(n)$: A274312.

$\approx 2.06372 \cdot 1.259305361.29232158^n$

1, 1, 1, 2, 2, 3, 4, 6, 7, 10, 13, 18, 23, 31, 40, 54, 69, 91, 118, 155, 199, 260, 334, 433, 555, 717, 917, 1180, 1506, 1929, 2458, 3140, 3990, 5081, 6445, 8185, 10361, 13125, 16581, 20956, 26424, 33322, 41940, 52782, 66312, 83293, 104467, 130979, ... , 4384627.
Finding small optimal regions

- Find all order ideals in $M(n)$ of sizes $s = 2, 4, 8, 16, 32, 64$.
- Calculate corresponding $F_S(t)$ polynomials.
- Compare all of them to find optimal regions.

- For $s = 2, 4, 8$ only projection is optimal.
- For $s = 16$: 5 optimal besides projection.
- For $s = 32$: 20 optimal besides projection.
- For $s = 64$: 56 optimal besides projection.

For all but 10 of the size 64, they are sets of minimal weight coset leaders of a linear code.
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## The terrible 10

### Table: Putative tiles

<table>
<thead>
<tr>
<th>$k$</th>
<th>$n$</th>
<th>generators of $V$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>12</td>
<td>${11}, {10, 5}, {9, 8}$</td>
</tr>
<tr>
<td>7</td>
<td>13</td>
<td>${12}, {10, 4}, {9, 8}$</td>
</tr>
<tr>
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<td>14</td>
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<td>15</td>
<td>${14, 1, 0}, {10, 2}$</td>
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<td>22</td>
<td>${21, 1}$</td>
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<td>23</td>
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<td>21</td>
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</table>
Tiles in $\mathbb{B}^n$

**Definition (Tile)**

A subset $S \subseteq \mathbb{B}^n$ is a *tile* if $\mathbb{B}^n$ is covered by disjoint translates of $S$.

$$\exists A \subseteq \mathbb{B}^n, \ A \oplus S = \mathbb{B}^n, \text{ uniquely.}$$

The set $A$ is called a *complement* of $S$. Note: $A$ is also a tile.

**Remark**

*This is equivalent to* $A \oplus S = \mathbb{B}^n$, $(A \oplus A) \cap (S \oplus S) = \{0\}$. 
Deciding if a subset is a tile

\[ \chi_S(x) = 1 \text{ if } x \in S, \ 0 \text{ otherwise.} \]
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- \( \chi_S(x) = 1 \) if \( x \in S \), 0 otherwise.
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- Integer program: Given a finite set of linear equalities and inequalities with integer variables, find values of variables satisfying all of them.
Necessary and Sufficient equations for a tile

Variables:

\[ z_u = \chi_A(u), \quad w_x = \widehat{\chi}_A(x). \]

Conditions:

\[ 0 \leq z_u \leq 1 \text{ and is an integer.} \]
\[ -|A| \leq w_x \leq |A| \text{ and is an integer.} \]
\[ w_0 = |A|. \]
\[ w_x = 0 \text{ if } x \neq 0 \text{ and } \widehat{\chi}_S(x) \neq 0. \]
\[ w_x = \sum_u (-1)^{x \cdot u} z_u \text{ for all } x. \]

Unfortunately too hard for CPLEX (high quality Integer Programming solver).
A relaxation

- Use \((A \oplus A) \cap (S \oplus S) = \{0\}\).
- Use that and equation of Hadamard transform: for 
  \(n = 12, 13, 14, 15\) sought for \(A\) doesn’t exist!
Necessary Equations for a tile

Variables:

\[ b_u = \chi_A \star \chi_A(u), \quad c_x = |\hat{\chi}_A(x)|^2. \]

Conditions:

\[ 0 \leq b_u \leq |A| \text{ and is an integer.} \]
\[ 0 \leq c_x \leq |A|^2 \text{ and is the square of an integer.} \]
\[ b_0 = |A|. \]
\[ c_0 = |A|^2. \]
\[ b_u = 0 \text{ if } u \neq 0 \text{ and } \chi_S \star \chi_S(u) \neq 0. \]
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A useful trick

- Equations for the Hadamard Transform involve $2^{2n}$ nonzero coefficients.
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- Sparse matrix multiplication: Calculate $Ax$ with $\#$ multiplications $= \#$ nonzero coefficients in $A$. 

Introduce extra variables for intermediate products.

Makes the problems for $n = 12, 13, 14, 15$ small enough for CPLEX. Others are still too big.
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- If we can write $A = B^{(1)} \cdots B^{(r)}$, and $B^{(i)}$ are sparse it’s a win.

Fast Hadamard Transform: $H = B^{(1)} \cdots B^{(n)}$, where $\#$ nonzeros in $B^{(i)}$ is only $2^{2n}$. 

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Pieces and Bins

- $X$: linear subspace of $\mathbb{B}^n$.
- Intersect $S$ with cosets of $X$: pieces.
- $\#((a \oplus S) \cap (b \oplus X)) = \#(S \cap ((a \oplus b) \oplus X))$.
- Must use all pieces to cover cosets of $X$.
- Can’t make it work for $n = 12, 13$ but can for all others.

<table>
<thead>
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<th>$k$</th>
<th>$r$</th>
<th>bin size</th>
<th>piece census</th>
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<tr>
<td>8</td>
<td>3</td>
<td>8</td>
<td>10<em>5, 1</em>6, 1*8</td>
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<tr>
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<td>3</td>
<td>8</td>
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<td>4</td>
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<td>21</td>
<td>2</td>
<td>4</td>
<td>15<em>2, 10</em>3, 1*4</td>
</tr>
</tbody>
</table>
Things to do

▶ Prove asymptotics for \# order ideals of a given size in $M(n)$. 

▶ Characterize those ideals yielding optimal regions.

▶ Better formulation for linear programming proof of non-tileability.

▶ When does bin packing work?

▶ Can we combine the two ideas?

▶ Ultimate goal: good characterization of those ideals yielding tiles.
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