Cyclic permutations, shuffles, and quasi-symmetric functions

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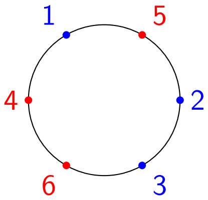

Based on joint work with

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Vic Reiner (Minnesota)
Yuval Roichman (Bar-Ilan)

Special thanks to Darij Grinberg (Drexel)

Outline

Permutations, shuffles, and descents

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- A = a finite set (alphabet) of size a

$$\begin{aligned} S_A &:= \text{the set of all permutations of } A \\ &= \text{bijections } u : [a] \to A \quad \text{(bijective words)} \end{aligned}$$

Example: $A = \{1, 3, 5, 7, 8\}, \qquad u = 51783 \in S_A$

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- A, B = disjoint finite sets; $u \in S_A, v \in S_B$

$$u \sqcup\!\sqcup v := \text{ the set of all shuffles of } u \text{ and } v$$

Example:
$A = \{1, 2, 3, 5\}, B = \{4, 6, 7\}, u = 1235 \in S_A, v = 764 \in S_B$

$$1723654 \in u \sqcup\!\sqcup v$$

Permutations, shuffles, and descents

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Question: What is the distribution of $\mathrm{des}(w)$ for $w \in u \sqcup\!\sqcup\, v$?

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In particular, what are the smallest and largest values of $\mathrm{des}(w)$?

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$$a = 4,\ b = 2,\ \mathrm{des}(u) = 2,\ \mathrm{des}(v) = 1$$

$$u \sqcup\!\sqcup\, v = \{143265,\ 143625,\ 146325,\ 164325,\ 614325,\ 143652,\ 146352,\ 164352,\ 614352,\ 146532,\ 164532,\ 614532,\ 165432,\ 615432,\ 651432\}$$

$$\sum_{w \in u \sqcup\!\sqcup\, v} q^{\mathrm{des}(w)} = 3q^2 + 9q^3 + 3q^4$$

Permutations, shuffles, and descents

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Theorem (Stanley '72)

If $|A| = a$, $|B| = b$ *with* $A \cap B = \varnothing$, *and* $u \in S_A$, $v \in S_B$ *with* $\mathrm{des}(u) = i$, $\mathrm{des}(v) = j$, *then the number of* $w \in u \sqcup\!\sqcup\, v$ *with* $\mathrm{des}(w) = k$ *is*

$$\binom{a+j-i}{k-i}\binom{b+i-j}{k-j}$$

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$u = 1432, \quad i = \mathrm{des}(u) = 2; \qquad v = 65, \quad j = \mathrm{des}(v) = 1$

$$\#\{w \in u \sqcup\!\sqcup v \,:\, \mathrm{des}(w) = k\} = \binom{3}{k-2}\binom{3}{k-1}$$

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- Actually holds on the level of descent sets.
- Follows from multiplication of quasi-symmetric functions.

Permutations, shuffles, and descents

Main Question:

What is the cyclic analogue?

The cyclic analogue

Cyclic permutations, shuffles, and descents

- $A =$ a finite set, $u \in S_A$. The cyclic permutation $[u]$ is the equivalence class (orbit) of u under cyclic shifts:

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Example: $u = 1234$, $v = 56789$

$$w = 734819562 \in u \sqcup\!\sqcup_c v$$

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Example: $u = 2\underset{\wedge}{4}15\underset{\wedge}{6}\underset{\wedge}{3} \in S_{[6]}$

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  Example: $v = 3\underset{\wedge}{4}15\underset{\wedge}{6}2 \in S_{[6]}$
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Introduced by Cellini ['95] (for arbitrary Weyl groups); further studied by Dilks, Petersen and Stembridge ['09] and others.

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Theorem (A-Gessel-Reiner-Roichman, $\sim$ 2021)

If $|A| = a$, $|B| = b$ with $A \cap B = \varnothing$, and $u \in S_A$, $v \in S_B$ with $\mathrm{cdes}([u]) = i$, $\mathrm{cdes}([v]) = j$, then the number of $[w] \in [u] \sqcup\!\sqcup_c [v]$ with $\mathrm{cdes}([w]) = k$ is

???

Cyclic quasi-symmetric functions

Symmetric functions

- A symmetric function is a formal power series $f \in \mathbb{Z}[[x_1, x_2, \ldots]]$ of bounded degree such that, for any $t \geq 1$, any two sequences $(i_1, \ldots, i_t)$ and $(i'_1, \ldots, i'_t)$ of distinct positive integers (indices), and any sequence $(m_1, \ldots, m_t)$ of positive integers (exponents), the coefficients of $x_{i_1}^{m_1} \cdots x_{i_t}^{m_t}$ and $x_{i'_1}^{m_1} \cdots x_{i'_t}^{m_t}$ in f are equal.

  Example:

$$x_1^4 x_2^2 x_3^5 + x_1^2 x_2^5 x_3^4 + x_1^5 x_2^4 x_3^2 +$$
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\begin{aligned}
&x_1^4x_2^2x_3^5 + x_1^2x_2^5x_3^4 + x_1^5x_2^4x_3^2 + \\
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&x_1^4x_2^2x_4^5 + x_1^2x_2^5x_4^4 + x_1^5x_2^4x_4^2 + \\
&x_1^4x_2^5x_4^2 + x_1^5x_2^2x_4^4 + x_1^2x_2^4x_4^5 + \ldots \in \mathrm{Sym}
\end{aligned}
$$

Quasi-symmetric functions

- A quasi-symmetric function is a formal power series $f \in \mathbb{Z}[[x_1, x_2, \ldots]]$ of bounded degree such that, for any $t \geq 1$, any two increasing sequences $i_1 < \ldots < i_t$ and $i'_1 < \ldots < i'_t$ of positive integers, and any sequence $(m_1, \ldots, m_t)$ of positive integers, the coefficients of $x_{i_1}^{m_1} \cdots x_{i_t}^{m_t}$ and $x_{i'_1}^{m_1} \cdots x_{i'_t}^{m_t}$ in f are equal.

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Example:

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Cyclic quasi-symmetric functions (new)

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Example:

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Comparison: similarities

- All are graded rings: Sym $\subsetneq$ cQSym $\subsetneq$ QSym.

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$$\begin{aligned}
\mathrm{Sym}_n &: \quad \{s_\lambda : \lambda \vdash n\} \quad \text{Schur functions} \\
\mathrm{QSym}_n &: \quad \{F_{n,J} : J \subseteq [n-1]\} \quad \text{Fundamental QSF} \\
\mathrm{cQSym}_n &: \quad \{\widehat{F}^{c}_{n,[J]} : \varnothing \ne J \subseteq [n] \text{ up to cyclic shifts}\} \\
& \quad \text{Normalized fundamental cQSF}
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- Dimension:

$$\begin{aligned} \dim \mathrm{Sym}_n &= p(n) \sim c^{\sqrt{n}} \quad \text{(partitions)} \\ \dim \mathrm{QSym}_n &= 2^{n-1} \quad \text{(compositions)} \\ \dim \mathrm{cQSym}_n &= \frac{1}{n}\sum_{d|n} \varphi(d) 2^{n/d} - 1 \sim \frac{1}{n} 2^n \end{aligned}$$

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- The involution ω:

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\begin{aligned}
\mathrm{Sym}_n: &\quad s_\lambda \leftrightarrow s_{\lambda'} \\
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$$\begin{aligned} \mathrm{Sym}_n : &\quad s_\lambda \leftrightarrow s_{\lambda'} \\ \mathrm{QSym}_n : &\quad F_{n,J} \leftrightarrow F_{n,[n-1]\setminus J} \\ \mathrm{cQSym}_n : &\quad \widehat{F}^c_{n,[J]} \leftrightarrow \widehat{F}^c_{n,[[n]\setminus J]} \end{aligned}$$

- Multiplication corresponds to (cyclic) shuffling. For $u \in S_A$, $v \in S_B$ ($A \cap B = \varnothing$, $A \cup B = C$):

$$F_{|A|,\mathrm{cDes}(u)} \cdot F_{|B|,\mathrm{cDes}(v)} = \sum_{w \in u \sqcup\!\sqcup v} F_{|C|,\mathrm{cDes}(w)}$$

$$F^c_{|A|,[\mathrm{cDes}(u)]} \cdot F^c_{|B|,[\mathrm{cDes}(v)]} = \sum_{[w] \in [u] \sqcup\!\sqcup_c [v]} F^c_{|C|,[\mathrm{cDes}(w)]}$$

Comparison: similarities and differences

- $s_{\lambda/\mu}$ is a linear combination, with nonnegative integer coefficients, of the basis elements of QSym; and similarly for cQSym, except when λ/μ is a connected ribbon!

$$s_{\lambda/\mu} = \sum_{T \in \mathrm{SYT}(\lambda/\mu)} F_{n,\mathrm{Des}(T)} \quad \text{[Gessel '84]}$$
$$= \sum_{[J]} m^c([J]) \widehat{F}^c_{n,[J]}$$

The latter follows from the existence of cyclic descents for standard Young tableaux (Rhoades ['10], A-Reiner-Roichman ['18], A-Elizalde- Roichman ['19], Huang ['20])

Comparison: differences

- The need for normalization: $\widehat{F}^c_{n,[J]} = \frac{1}{d_J} F^c_{n,J}$, where

$$d_J := |Stab_{\mathbb{Z}/n\mathbb{Z}}(J)| = \#\{i \in \mathbb{Z}/n\mathbb{Z} \,:\, J + i \equiv J \pmod{n}\}$$

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- The "non-Escher" property: clearly

$$\mathrm{cDes}(u) \neq \varnothing, [n] \qquad (\forall u \in S_n)$$

but $\widehat{F}^c_{n,[\varnothing]} = h_n = s_{(n)}$ and $\widehat{F}^c_{n,[[n]]} = e_n = s_{(1^n)}$ are important symmetric functions which should be part of the family.

Other ingredients

An unusual ring homomorphism

- Define a new product on $\mathbb{Z}[[q]]$ by

$$q^i \odot q^j := q^{\max(i,j)},$$

with the usual addition, to get the ring $\mathbb{Z}[[q]]_\odot$.

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- Define a ring homomorphism $\Psi : \mathbb{Z}[[\mathbf{x}]]_{\mathrm{bd}} \to \mathbb{Z}[[q]]_\odot$ by
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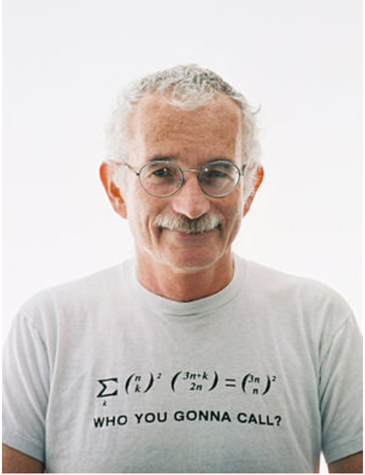
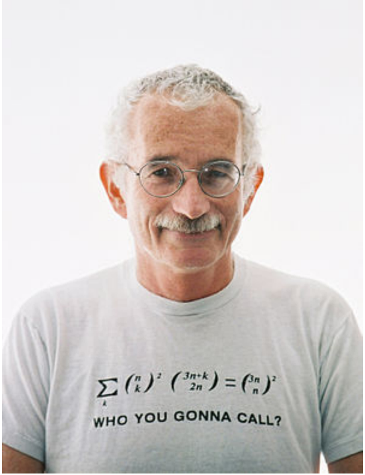
$$\Psi(F_{n,J}) = \frac{q^{|J|+1}}{(1-q)^n} \qquad (J \subseteq [n-1])$$


$\sum_k \binom{n}{k}^2 \binom{3n+k}{2n} = \binom{3n}{n}^2$
WHO YOU GONNA CALL?

A triple binomial identity



Doron Zeilberger

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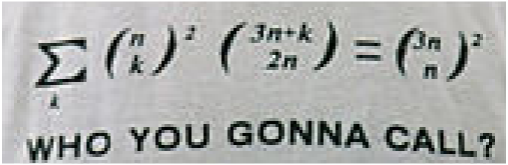

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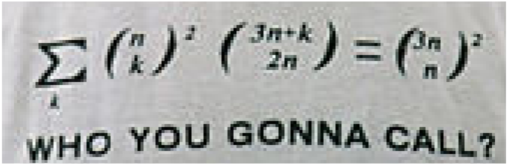

This is a special case of the triple-binomial identity

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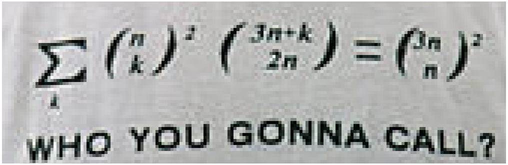

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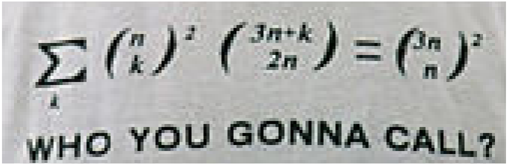

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We need the general case.

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- Stanley used a q-analogue of the above identity, proved by Gould (1972), and equivalent to one by Jackson (1910). Gould's proof was a variation of one by Nanjundiah (1958) of the $q = 1$ case.

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Theorem (A-Gessel-Reiner-Roichman, ∼ 2021)

If $|A| = a$, $|B| = b$ *with* $A \cap B = \varnothing$, *and* $u \in S_A$, $v \in S_B$ *with* $\mathrm{cdes}([u]) = i$, $\mathrm{cdes}([v]) = j$, *then the number of* $[w] \in [u] \sqcup\!\sqcup_c [v]$ *with* $\mathrm{cdes}([w]) = k$ *is*

$$\frac{k(a-i)(b-j) + (a+b-k)ij}{(a+j-i)(b+i-j)} \binom{a+j-i}{k-i} \binom{b+i-j}{k-j}$$

Summary

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- The ring cQSym of cyclic quasi-symmetric functions is intermediate between Sym and QSym.
- It has many properties in common with QSym, but also some interesting unique features.
- It has applications to combinatorial enumeration (and to other areas).

Thank You!