Noncommutative Catalan numbers: Orthogonality, Positivity and Beyond

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Our goal is threefold:

- Further generalize noncommutative Catalan numbers from our previous paper and, by specializing them, obtain (commutative and noncommutative) deformations of several classical sequences.

- Using these generalized Catalan numbers as (commutative or noncommutative) moments, complete the theory of noncommutative orthogonal polynomials which originated in 1994 by Gelfand, Krob, Lascoux, Leclerc, Retakh, Thibon.

- Establish noncommutative total positivity of the corresponding Hankel matrices.

cf. V.I. Lenin, "The Three Sources and Three Component Parts of Marxism"

Three classical definitions of Catalan numbers:

• c_n is the number of all monotonic lattice paths in $[0,n]\times[0,n]$ from $(0,0)$ to (n,n) which lie below the diagonal.

• Set

$$d_m(n) := \begin{vmatrix} c_m & c_{m+1} & \dots & c_{m+n} \\ c_{m+1} & c_{m+2} & \dots & c_{m+n+1} \\ & \dots & \dots & \\ c_{m+n} & c_{m+n+1} & \dots & c_{m+2n} \end{vmatrix} .$$

Catalan numbers are solutions of the equations $d_m(n) = 1$ for $m = 0, 1$ and $n \geq 0$.

• Let

$$P = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & \ddots \\ 0 & 1 & 2 & 1 & \ddots \\ 0 & 0 & 1 & 2 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Then $(P^n)_{00} = c_n = \frac{2n!}{n!(n+1)!}$, the n-th Catalan number.

Remark: Another way

$$P' = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & \ddots \\ 1 & 1 & 1 & 1 & \ddots \\ 1 & 1 & 1 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Then $((P')^n)_{00} = c_n$.

One can also obtain many sequences from OEIS for various matrices P.

Noncommutative Version

Let F be the free group generated by x_k, $k \in \mathbf{Z}_{\geq 0}$ and F_m be the (free) subgroup of F generated by $x_0, \ldots, x_m$.

To each point $p = (p_1, p_2)$ on a plane we associate its *content* $c(p) := p_1 - p_2$. If P is a Catalan path and $p \in P$ then $c(p) \geq 0$.

We say that a point $p = (p_1, p_2)$ is a *south-east (resp. northwest) corner* of a path P if $(p_1 - 1, p_2) \in P$ and $(p_1, p_2 + 1) \in P$ (resp. $(p_1, p_2 - 1) \in P$ and $(p_1 + 1, p_2) \in P$).

To each Catalan path P from $(0,0)$ to (n,n) we assign an element $M_P \in F_n$ by

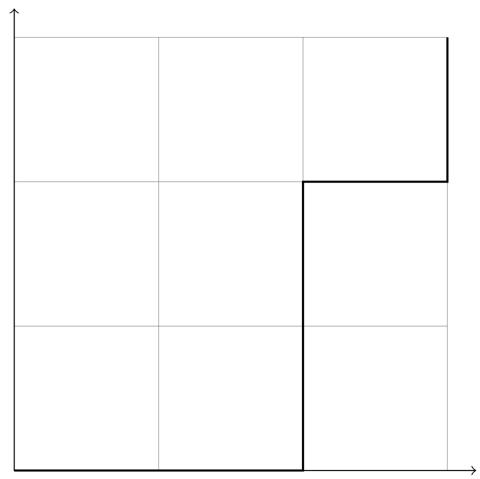
$$M_P = \prod x_{c(p)}^{\sigma(p)} ,$$

where the product is over all corners $p \in P$ (taken in the natural order) and

$$\sigma(p) = \begin{cases} 1 & \text{if } p \text{ is southeast} \\ -1 & \text{if } p \text{ is northwest} \end{cases} .$$

Let $\mathcal{P}_n$ be the set of all Catalan paths from $(0,0)$ to (n,n). Define the *noncommutative Catalan number* $C_n \in \mathbf{F}_n$ by

$$C_n = \sum_{P \in \mathcal{P}_n} M_P .$$



Example: $M_P = x_2 x_0^{-1} x_1$ for the above path P

$$C_0 = x_0,\ C_1 = x_1,$$
$$C_2 = x_2 + x_1 x_0^{-1} x_1,$$
$$C_3 = x_3 + x_2 x_1^{-1} x_2 + x_2 x_0^{-1} x_1 + x_1 x_0^{-1} x_2 + x_1 x_0^{-1} x_1 x_0^{-1} x_1$$

Under the counit homomorphism $x_k \mapsto 1$ the image of C_n is c_n, the ordinary Catalan number.

Symmetry: Under anti-automorphism $\bar{\cdot}$ of $\mathbf{Z}F$ such that $\overline{x}_k = x_k$ for $k \geq 0$ we have $\overline{C}_n = C_n$ for any n.

Quasideterminant equations

Introduce Hankel matrix

$$H_m(n) = \begin{pmatrix} C_m & C_{m+1} & \dots & C_{m+n} \\ C_{m+1} & C_{m+2} & \dots & C_{m+n+1} \\ & \dots & \dots & \\ C_{m+n} & C_{m+n+1} & \dots & C_{m+2n} \end{pmatrix}$$

Define its quasideterminant $Q_m(n)$ as the inverse to the southeast element of $H_m(n)^{-1}$. Can be computed as

$$Q_m(n) = C_{m+2n} - r_n(m) H_m(n-1)^{-1} r_n(m)^T$$

where $r_n(m) = (C_{m+n}, \dots C_{m+2n-1})$.

Note: $Q_m(n) = d_m(n)/d_m(n-1) = 1$ under specialization $x_k \mapsto 1$.

Theorem 1. Laurent polynomials C_n are solutions of the system

$$Q_m(n) = x_{m+2n}, \quad m = 0, 1; \quad n \geq 0$$

LDU-factorizations of Hankel matrices

Let $H_m := H_m(\infty)$.

Problem: $H_m = L_m D_m U_m$. Describe entries of L_m, D_m, U_m for $m = 0, 1$. It is clear that $U_m(i,j) = \overline{L}_m(j,i)$.

$$D_m = \mathrm{diag}(x_m, x_{m+2}, x_{m+4}, \dots)$$

To describe entries of L_m introduce *truncated Catalan numbers (parking functions).*

Let $\mathcal{P}_n^k$ be the set of all Catalan paths P from $(0,0)$ to (n,n) such that the rightmost southeast corner of P has coordinates (n,s), $s \le k \le n$. Then

$$C_n^k = \sum_{P \in \mathcal{P}_n^k} M_P$$

$C_n^0 = x_n$, $C_n^1 = x_n + \sum_{i=1}^{n-1} x_i x_{i-1}^{-1} x_{n-1}$, $C_n^{n-1} = C_n^n = C_n$

Theorem 2. For $j \ge i$ and $m = 0, 1$

$$L_m(j,i) = C_{j+i+m}^{j-i} \cdot x_{2i+m}^{-1}$$

In particular, $L_m(j,0) = C_{j+m} \cdot x_m^{-1}$

Entries of L_m^{-1} are *noncommutative binomial coefficients* (up to a sign).

$y_k := x_k x_{k-1}^{-1}$; $\quad y_J := y_{j_k+k-1} \cdots y_{j_2+1} y_{j_1}$
for any $J = \{j_1 < j_2 < \cdots < j_k\}$, $k \geq 1$.

Define the binomial coefficient

$$\binom{\mathbf{n}}{\mathbf{k}} = \sum_{J \subset [1,n], |J|=k} y_J$$

Under specialization $x_k \mapsto q^{k(k-1)/2}$ we have

$$\binom{\mathbf{n}}{\mathbf{k}} \mapsto q^{k(k-1)} \binom{n}{k}_q$$

Examples: $\binom{\mathbf{n}}{\mathbf{0}} = 1$, $\binom{\mathbf{n}}{\mathbf{1}} = \sum_{i=1}^{n} y_i$,
$\binom{\mathbf{n}}{\mathbf{2}} = \sum_{1 \leq i < j \leq n} y_{j+1} y_i$, $\binom{\mathbf{n}}{\mathbf{n}} = y_{2n-1} \ldots y_3 y_1$

Pascal: $\binom{\mathbf{n+1}}{\mathbf{k}} = \binom{\mathbf{n}}{\mathbf{k}} + y_{n+k} \binom{\mathbf{n}}{\mathbf{k-1}}$

Theorem 3. For $m = 0, 1$ and $0 \leq i \leq j$

$$L_m^{-1}(j, i) = (-1)^{i+j} \binom{\mathbf{i+j+m}}{\mathbf{j-i}}$$

Third approach

Theorem 4. Let $\mathbf{x} = \{x_0, x_1, \ldots\}$ be a sequence of free variables. Set

$$J_{\mathbf{x}} = \begin{pmatrix} x_1x_0^{-1} & 1 & 0 & \cdots \\ x_2x_0^{-1} & x_2x_1^{-1} + x_3x_2^{-1} & 1 & \ddots \\ 0 & x_4x_2^{-1} & x_4x_3^{-1} + x_5x_4^{-1} & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

Then $(J_{\mathbf{x}})^n_{00} \cdot x_0 = C_n$.

To put expressions $(P^n)_{00}$ in a context. Set

$$E = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ & & \cdots & & \end{pmatrix}$$

Consider matrix equation

$$L_P P = E L_P$$

P is *production matrix*,
L_P is *output matrix*.
Exercise. The i-th row of L is the 0-th row of P^i, $i \geq 0$.

If P has $P_{i,i+1} = 1$ and P is a Hessenberg matrix, i.e. $P_{ij} = 0$ for all $j > i+1$ then L_P is lower unitriangular and thus invertible.

A symmetric version of $J_{\mathbf{x}}$:

$$\tilde{J}_{\mathbf{x}} = \begin{pmatrix} x_1 & x_2 & 0 & \dots \\ x_2 & x_2 x_1^{-1} x_2 + x_3 & x_4 & \ddots \\ 0 & x_4 & x_4 x_3^{-1} x_4 + x_5 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

Then $H_1 = L_0 \tilde{J}_{\mathbf{x}} \overline{L}_0$.

Fix a unital ring R and an anti-involution $\bar{\cdot}$ on R. For any left R-linear map $\mu : R[t] \to R$ define an inner product $\langle \cdot, \cdot \rangle_\mu$ on $R[t]$ by

$$\langle p, \sum_i a_i t^i \rangle_\mu := \sum_i \mu(p \cdot t^i) \overline{a}_i$$

For any matrix $M \in Mat(R)$ denote by M^T the transposed of M, i.e., $(M^T)_{ij} = \overline{M}_{ji}$ for all $i, j \geq 0$. We say that M is *symmetric* if $M^T = M$, i.e., $M_{ji} = \overline{M}_{ij}$ for all $i, j \geq 0$.

Theorem 5. Let $\pi_\bullet = \{\pi_k\}$ be a monic orthogonal basis in $R[t]$ with respect to some $\langle\cdot,\cdot\rangle_\mu$. Then:

(a) $\pi_\bullet$ admits a tri-diagonal Jacobi matrix of $\pi_\bullet$ (i.e., $t\pi_k = \sum_\ell J_{k\ell}\pi_\ell$) such that JD is symmetric in $Mat(R)$ with a unique symmetric diagonal matrix D, $D_{00} = \mu_0$.

(b) Let $H_k := \big(\mu(t^{k+i+j})\big)$, $J_k = \big(\langle\pi_i(t), t^k\pi_j(t)\rangle_\mu\big)$, $i, j, k \geq 0$. Then $H_k = L_J J_k L_J^T$, $J_k = J^k D$ for $k \geq 0$.

(c) $\langle\pi_k(t), \pi_k(t)\rangle_\mu = D_{kk}$ and $\pi_k(t) = \sum_{\ell=0}^{k}(L_J^{-1})_{k\ell}\cdot t^\ell$ for $k \geq 0$.

Corollary. $\mu(t^n) = (J^n D)_{00}$ and

$$H_0 = L_J D L_J^T,\ \ J = L_J^{-1} H_1 L_J^{-T} D^{-1}\ .$$

This simplifies the parametrization of (commutative and noncommutative) orthogonal polynomials because the entries of the matrix $L_{J_{\mathbf{x}}}^{-1}$ involved in the Theorem are, up to sign, are generalized binomial coefficients, in particular

$$\pi_n(t) = \sum_{k=0}^{n} (-1)^{n-k} \binom{\mathbf{n}+\mathbf{k}}{\mathbf{n}-\mathbf{k}} t^k$$

for all $n \geq 0$. The polynomials $\pi_\bullet$ satisfy the recursion $\pi_0(t) = x_0$, $\pi_1(t) = t - x_1$,

$$t\pi_n(t) = \pi_{n+1} + (x_{2n+1}x_{2n}^{-1} - x_{2n}x_{2n-1}^{-1})\pi_n(t) + x_{2n}x_{2n-2}^{-1}\pi_{n-1}(t),\ n \geq 1\ .$$

Specializing all x_i to 1, $\pi_\bullet$ becomes the Chebyshev polynomials of third kind, the polynomials satisfy the recursion $\pi_0 = 1$, $\pi_1 = t - 1$,

$$t\pi_n = \pi_{n-1} + 2\pi_n + \pi_{n+1},\ n \geq 1$$

Returning to the general case, we can notice some positivity built in $J_{\mathbf{x}}$, which is manifested by its factorization:

$$J_{\mathbf{x}} = \begin{pmatrix} 1 & 0 & 0 & \dots \\ y_2 & 1 & 0 & \ddots \\ 0 & y_4 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \cdot \begin{pmatrix} y_1 & 1 & 0 & & \dots \\ 0 & y_3 & 1 & 0 & \ddots \\ 0 & 0 & y_5 & 1 & \ddots \\ \vdots & \ddots & \ddots & & \ddots \end{pmatrix},$$

where we abbreviated $y_i = x_i x_{i-1}^{-1}$ for $i \geq 1$. Thus, $J_{\mathbf{x}}$ *totally nonnegative* if all x_i are declared positive.