Talk plan

1. continued fractions
2. 4-color theorem
3. The Eisenstein integers
4. Triangulations w/degrees 6,...,6,2,2,2
5. Triangulations w max degree 6
Continued Fractions:

\[
\frac{3}{7} = \frac{1}{\frac{7}{3}} = \frac{1}{2 + \frac{1}{3}}
\]

\[
\frac{7}{10} = \frac{1}{\frac{10}{7}} = \frac{1}{\frac{1}{1 + \frac{3}{7}}} = \frac{1}{\frac{1}{2 + \frac{1}{3}}}
\]
geometric view of cont. fractions.

3/7

7/10
tree view of continued fractions

\[
\begin{align*}
\frac{a}{a+b} & & \frac{b}{a+b} \\
\begin{array}{c}
\frac{a}{b}
\end{array}
\end{align*}
\]

(slow Gauss map)

\[
\begin{array}{cccc}
7/10 & 3/10 \\
3/7 & 4/7 \\
1/3 & 1/2 \\
1/1 & 2/3 \\
3/8 & 5/8 \\
4/5 & 1/5
\end{array}
\]
4-Color theorem (popular version)

The regions of a planar graph can be colored with 4 colors so that adjacent regions get different colors.

computer assisted proof by Haken and Appel (late 70s)

Improved by Robertson and Seymour (90s)

Proof checked (?) recently by an automated checker.
4-Color theorem (dual version):

The vertices of a planar graph can be colored with 4 colors so that adjacent vertices get different colors. (It suffices to prove this for triangulations.)
Proof of the 6 color theorem..

1. Find a vertex of degree <6 and remove it.
2. Color the rest, by induction.
3. Put the vertex back in and use the left over color.
Proof of the 5 color theorem (Heawood):..
4-Color theorem (geometric version):

The faces of a triangulation can be colored black and white so that around each vertex #B and #W are congruent mod 3.

Geometrically, this is equivalent to the existence of a nowhere collapsing simplicial map to a tetrahedron.
A halloween themed example
Eisenstein integers

\[ a = \exp(\pi i/6) \]

\[ p + qa \]

\[ \frac{p}{q} \]
This is a coloring of \( T(2/3) \) with degree sequence 6,\ldots,6,2,2,2,2 + 3a.
This is a trapezoid necklace.
A big necklace entails a smaller one.

\[
\frac{3}{5} \quad \frac{2}{3}
\]
2 big ones entail the same small one.

This operation does the slow Gauss map!
The empty 3/5-flower
The empty $3/7$-flower

$$3/7 = \frac{1}{2 + \frac{1}{3}}$$
The 3/5-flower
The 3/5-coloring
Properties:

Let \( f = \) number of B/W edges:

- if \( \{ p_n / q_n \} \) has an irrational limit then \( f_n / F \) converges to 0.

- if \( \{ p_n / q_n \} \) are the continued fraction approximants of a quadratic irrational, \( f_n^2 / F \) has a well-defined limit \( \eta \).

Def: \( \eta \) is the isoperimetric ratio of the quadratic irrational.
Sample calculations

$$\eta(\phi^{-1}) = \phi^6, \quad \phi = \frac{\sqrt{5} + 1}{2}.$$ 

Let $\psi_n = \sqrt{n} - \text{floor}(\sqrt{n})$.

$$\eta(\psi_2) = \frac{75 + 53\sqrt{2}}{7} \quad \eta(\psi_3) = \frac{132 + 72\sqrt{3}}{13}$$

$$\eta(\psi_5) = \frac{321 + 137\sqrt{5}}{19} \quad \eta(\psi_6) = \frac{27 + 9\sqrt{6}}{2}$$

$$\eta(\psi_7) = \frac{3100 + 856\sqrt{7}}{259} \quad \eta(\psi_8) = \frac{1569 + 370\sqrt{8}}{98}$$
Triangulations of max degree 6.

If you build the triangulation out of equilateral triangles, the resulting metric space is isometric to the flat cone manifold with at most 12 cone points of positive curvature.

Examples:
Thurston coordinates

- Independent variables
- Dependent variables
- Origin

Gives local coords from the moduli space to $\mathbb{C}^4$. Triangs correspond to $(EIS)^4$.
The area of the flat cone structure in Thurston coords is the diagonal part of a Hermitian form of signature \((1,n)\).

It means that globally the triangulations are "rational points" in a complex hyperbolic orbifold.
Experimental observation.

Call a 4-coloring solution **generic** if you never see an alternating pattern around a vertex:

Call 2 generic solutions **equivalent** if they define the same shape packing up to homeomorphism.

The equivalence classes of generic solutions seem to be rational linear subspaces of the moduli space, having half the dimension of the total space.
Here's what rational subspace means:

Compare Tutte's square packing theorem.