Finding the jewel in the lotus

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Abstract

There are various graphs defined on groups which the group structure in some way. The most famous is the *commuting graph*, where two elements are joined if they commute; others include the *power graph* (two elements joined if one is a power of the other), the *enhanced power graph* (two elements joined if both are powers of a single element), and the *generating graph* (two elements joined if they generate the group).

The alternating group A_5 is the smallest non-abelian finite simple group, having order 60. A few years ago, Colva Roney-Dougal and I were rather shocked to discover that the automorphism group of its generating graph has order 23482733690880. The commuting graph is even more extreme: it has 4770901323934635707759680000 automorphisms. Usually, a large automorphism group indicates a beautiful graph, but here these large groups are mostly rubbish, which has to be stripped away.

The way to do this is *twin reduction*, identifying pairs of vertices with the same neighbours apart perhaps from one another. It is known that the result, up to isomorphism, is independent of the way the reduction is done, and is a single vertex if and only if the original graph has no induced path of length 4. For each type of graph, there is a problem of deciding for which groups the graph has this property.

Occasionally we strike lucky, and the resulting graph has very nice properties. For one example, if the group is the Mathieu group M_{12} of order 7920, and the edge set of the graph is the difference of the edge sets of the power graph and enhanced power graph, we find a connected bipartite graph on $165 + 220$ vertices, with valencies 4 and 3, with diameter and girth 10 and automorphism group M_{11}.