Combinatorics behind the degenerate Eulerian numbers

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Permutations: definitions

Definition

Permutation of a length n is an ordering of the elements of the set $[n] = \{1, 2, \cdots, n\}$.

For any $\pi \in S_n$

$$asc(\pi) + des(\pi) = n - 1$$

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Let $\pi = \pi_1 \pi_2 \cdots \pi_n$ be a permutation. The index $k < n$ is

- an ascent of π if $\pi_k < \pi_{k+1}$ ($\sim$ pair π_k, π_{k+1} is a rise)
- a descent of π if $\pi_k > \pi_{k+1}$ ($\sim$ pair π_k, π_{k+1} is a fall)

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Example

Let $\pi = 31452$.

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Generating functions

$$\frac{t-1}{t-e^{u(t-1)}} = \sum_{n\geq 0}\sum_{k\geq 0} A(n,k)t^k \frac{u^n}{n!},$$

$$\frac{A_n(t)}{(1-t)^{n+1}} = \sum_{j\geq 0}(j+1)^n t^i.$$

Different notations for the Eulerian polynomial

Today:

$$\frac{1-t}{1-te^{u(1-t)}} = \sum_{n\geq 0}\sum_{k\geq 0} A(n,k)t^k \frac{u^n}{n!},$$

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$$A_1(t) = t,$$
$$A_2(t) = t^2 + t,$$
$$A_3(t) = t^3 + 4t^2 + t,$$
$$A_4(t) = t^4 + 11t^3 + 11t^2 + t,$$

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Generalizations: q-calculus

q-integers:

$$[n]_q := \frac{1-q^n}{1-q} = \sum_{j=0}^{n-1} q^j$$

Carlitz q-Eulerian pol. (1954)

$$\frac{A_n(t,q)}{(t;q)_{n+1}} = \sum_{j\geq 0} t^j ([j+1]_q)^n$$

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$$A_n(t,q) = \sum_{\pi\in S_n} t^{|des(\pi)|}q^{maj(\pi)}, \qquad \text{where } maj(\pi) := \sum_{\pi_i>\pi_{i+1}} i$$

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- $1+2=3.$

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Generalized Eulerian numbers [Carlitz, Scoville 1974]

$A_{n,k}$ - # of ascending runs

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Symmetry relation:

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NOTE:

polynomial in α, β with positive integer coef.

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$$A(r,s|\alpha,\beta) = (r+\beta)A(r,s-1|\alpha,\beta) + (s+\alpha)A(r-1,s|\alpha,\beta)$$

Combinatorial applications of $A(r, s|\alpha, \beta)$

Generalization of

rises

falls

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β # of symbols $0'$

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Construction: initial step ($n = 1$)

$$\underbrace{0\ldots0}_{\alpha}\,1\,\underbrace{0'\ldots0'}_{\beta}$$

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rises a pair of consecutive elements a, b with $a < b$, where a may be 0

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falls a pair of consecutive elements a, b with $a > b$, where b may be $0'$

maxima the element b is a maximum if a, b, c are consecutive and a, b is a rise while b, c is a fall.

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Combinatorial applications of $A(r, s|\alpha, \beta)$

Construction: requirement

Symbols $2, 3, \ldots, n$ can be inserted in all possible places w.r.t the requirements

1. there is at least one 0 on the extreme left
2. there is at least one $0'$ on the extreme right

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Example (Carlitz, Scoville 1974)

$$\underbrace{0\ldots0}_{\alpha}\,1\,\underbrace{0'\ldots0'}_{\beta}$$

- 1 rise
- 1 fall
- 1 maximum

- # of rises $r = 4$
- # of falls $s = 3$
- # of maxima $k = 1$

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Example (Carlitz, Scoville 1974)

$$\underbrace{0\ldots0}_{\alpha}1\underbrace{0'\ldots0'}_{\beta}$$

$02301540'0'60'$

- 1 rise
- 1 fall
- 1 maximum

- # of falls $s = 3$
- # of maxima $k = 1$

Combinatorial applications of $A(r, s|\alpha, \beta)$

Construction: requirement

Symbols $2, 3, \ldots, n$ can be inserted in all possible places w.r.t the requirements

1. there is at least one 0 on the extreme left
2. there is at least one $0'$ on the extreme right

Example (Carlitz, Scoville 1974)

$$\underbrace{0\ldots0}_{\alpha}1\underbrace{0'\ldots0'}_{\beta}$$

- 1 rise
- 1 fall
- 1 maximum

$$02301540'0'60'$$

- $\alpha = 2$
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$$P(r, s|\alpha, \beta) = (r + \beta - 1)P(r, s - 1|\alpha, \beta) + (s + \alpha - 1)P(r - 1, s|\alpha, \beta)$$

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$$P(r + 1, s + 1|\alpha, \beta) = A(r, s|\alpha, \beta)$$

Carlitz degenerate Eulerian polynomials

Degenerate exponential function of Carlitz, 1956

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$$\frac{1-t}{1-t(1+\lambda(u-tu))^{\mu}} = 1 + \sum_{n=1}^{\infty} \frac{u^n}{n!} \sum_{k=1}^{n} A_{n,k}(\lambda) t^k.$$

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$$A_0 = 1,$$
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NOTE:

$$\sum_{k=1}^{n} A_{n,k}(\lambda) = n!$$

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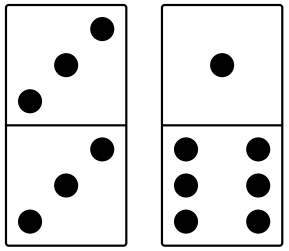
> The numbers $S(n,k)$, $S_1(n,k)$, B_n, $A_{n,k}$ have various arithmetic and combinatorial properties. We do not consider any such questions for the quantities discussed in the present paper but hope to do so in a later communication.

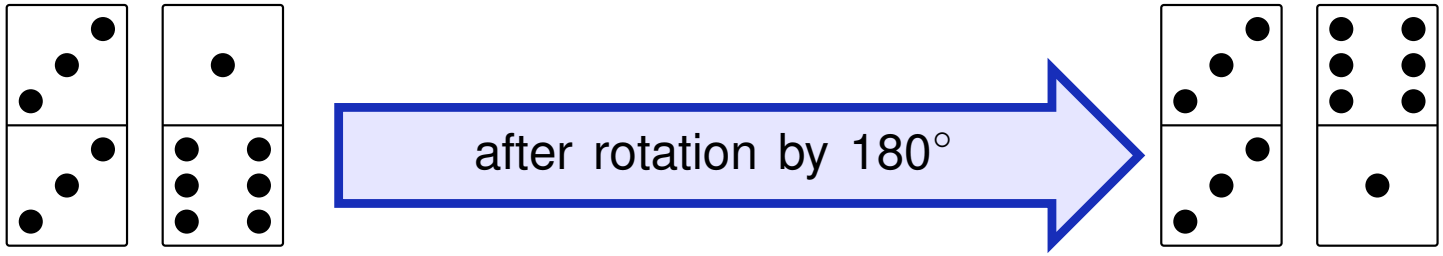
Generalization of permutations



The permutation is not changed. Ascents and descents are not changed.

Let's change it! How?

Generalization of permutations



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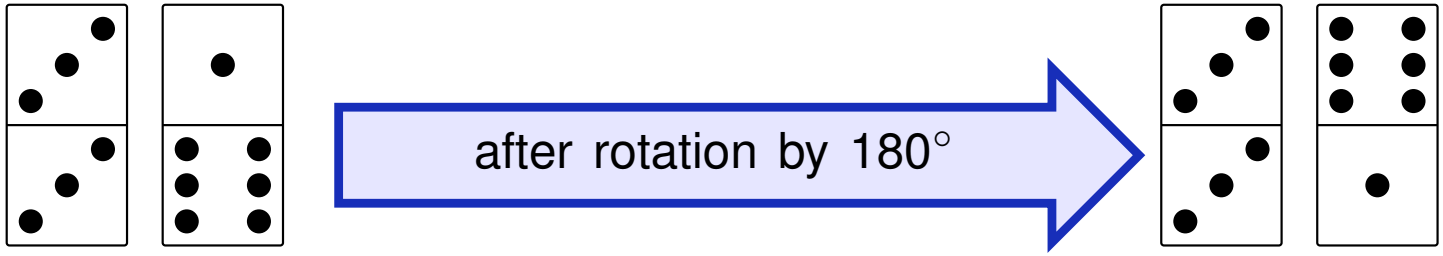
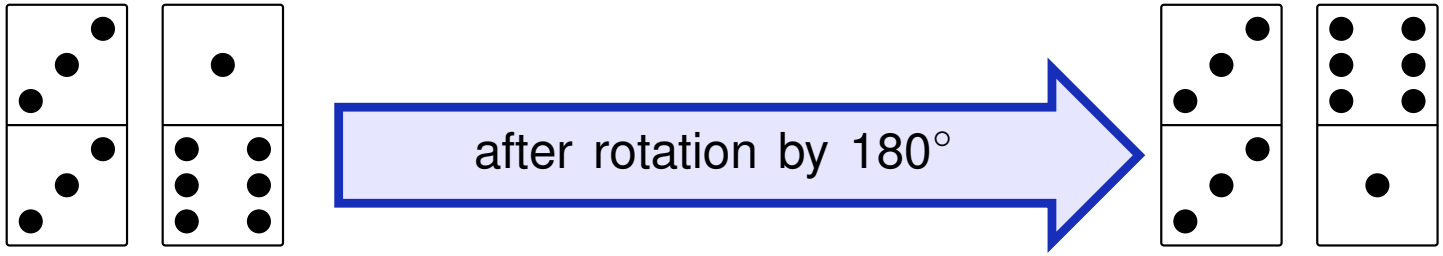


Let us illustrate a permutation $\pi = 31452$ in a domino tiles style:

Generalization of permutations



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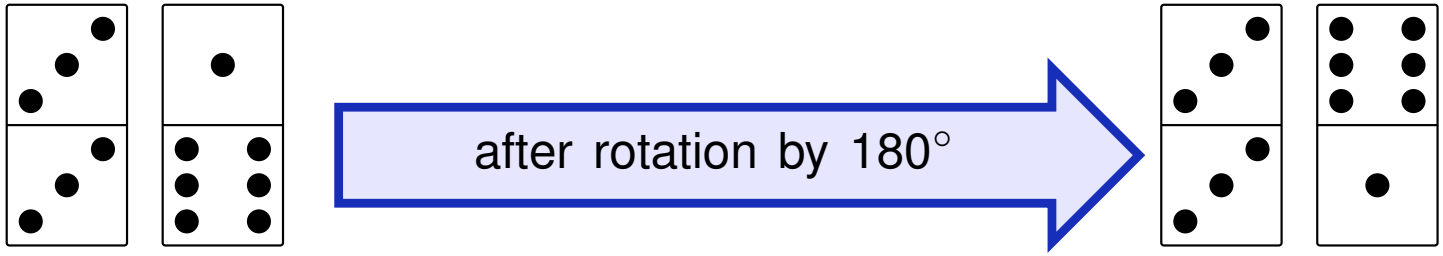

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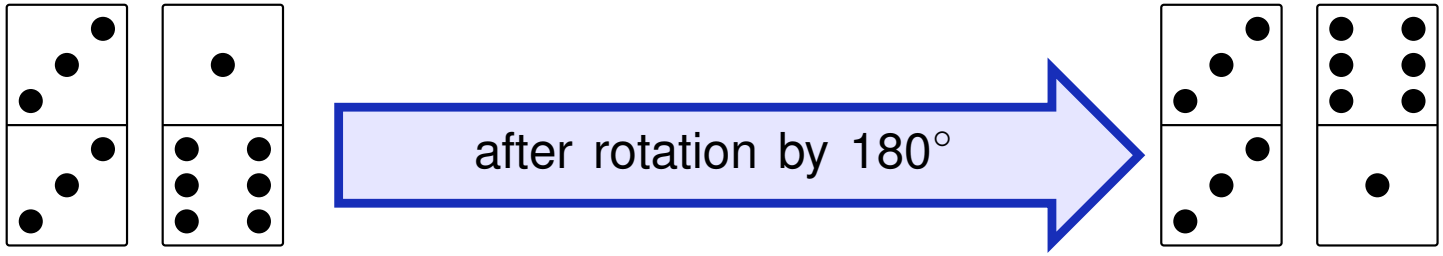

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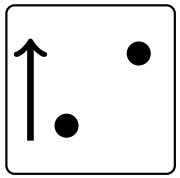
Generalized permutations

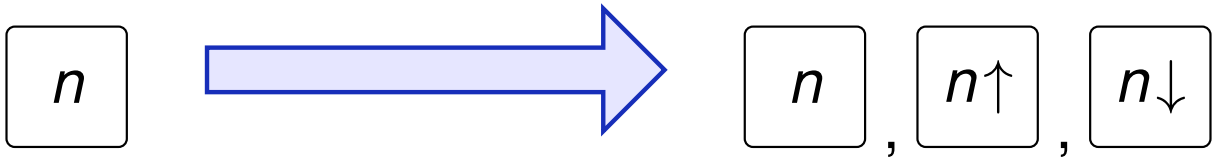
More tiles:



Generalized permutations

More tiles:

For each $n \in \mathbb{N}$



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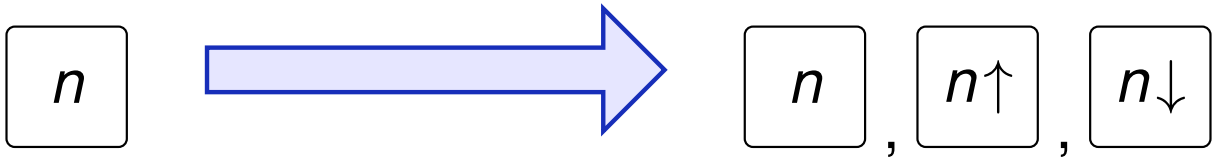

Number of permutations:

$$n! \quad \Longrightarrow \quad (3n)!!! \quad (= 3^n n!)$$

where $(3n)!!! = \prod_{i=1}^{n}(3i)$. **Triple permutations**

Ascents

Example (Ascents in classical permutation)

Let $\pi = 31452$. Then

- $asc(\pi) = \{2, 3\}$
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Upward arrow tile adds an ascent!

We insert an upward arrow tile $\boxed{n\uparrow}$ and a downward arrow tile $\boxed{n\downarrow}$ on the left from permutation tiles.

Generalized permutations: construction

Recursive construction of $\pi' = \pi'_1 \cdots \pi'_n$ from $\pi = \pi_1 \cdots \pi_{n-1}$

4. $\pi_1 \cdots \pi_i \boxed{n} \pi_{i+1} \cdots \pi_{n-1}$, where $i \in des(\pi)$,
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Generalized permutations with ascents: construction

Example

n=1 $\boxed{1}$

n=3 Possible tiles for insertion: $\boxed{3}$, $\boxed{3\uparrow}$, $\boxed{3\downarrow}$

Generalized permutations with ascents: construction

Example

n=1 $\boxed{1}$

n=2 Possible tiles for insertion: $\boxed{2}$, $\boxed{2\uparrow}$, $\boxed{2\downarrow}$

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Permutations: $\boxed{1}\,\boxed{2}$, $\boxed{2}\,\boxed{1}$, $\boxed{2\uparrow}\,\boxed{1}$, $\boxed{2\downarrow}\,\boxed{1}$

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Permutations:

$\boxed{1}\ \boxed{2} \Rightarrow \boxed{3\uparrow}\ \boxed{1}\ \boxed{2}$, $\boxed{1}\ \boxed{3\uparrow}\ \boxed{2}$, ...

Generalized permutations with ascents: construction

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Generalized permutations: construction

Example (Legal recursive construction)

Let us consider a recursive construction of a permutation $\pi = 2\downarrow, 4, 3\uparrow, 1$.

4: $\pi = 2\downarrow, 4, 3\uparrow, 1$. In accordance with the Rule3 $asc(\pi) = \{1, 3\}$ and $|asc(\pi)| = 2$.

Example (Illegal construction)

Let us consider another permutation $\pi = 2\downarrow, 1, 3\uparrow, 4$.

Step 2: $\pi = 2\downarrow, 1, 3\uparrow, 4$.

Generalized permutations: construction

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Let us consider a recursive construction of a permutation $\pi = 2\downarrow, 4, 3\uparrow, 1$.

1: a trivial permutation $\pi = 1$ with no ascents.

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Let us consider a recursive construction of a permutation $\pi = 2\downarrow, 4, 3\uparrow, 1$.

1: a trivial permutation $\pi = 1$ with no ascents.

$2\downarrow$: $\pi = 2\downarrow, 1$ and w.r.t the Rule 6 the number of ascents is not changed.

Generalized permutations: construction

Example (Legal recursive construction)

Let us consider a recursive construction of a permutation $\pi = 2\downarrow, 4, 3\uparrow, 1$.

- 1: a trivial permutation $\pi = 1$ with no ascents.
- $2\downarrow$: $\pi = 2\downarrow, 1$ and w.r.t the Rule 6 the number of ascents is not changed.
- $3\uparrow$: $\pi = 2\downarrow, 3\uparrow, 1$. In accordance with the Rule5 $asc(\pi) = \{2\}$ and $|asc(\pi)| = 1$.

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- $3\uparrow$: $\pi = 2\downarrow, 1, 3\uparrow$. ILLEGAL w.r.t. the Rule5

Generating function

Notations:

- GSA_n – a set of all generalized permutations of $[n]$ with ascents,
- $\pi = \pi_1\pi_2\ldots\pi_n \in \mathrm{GSA}_n$,
- $\mathrm{nua}(\pi)$ – a number of tiles with $\uparrow$ in π,
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We denote by $A(n,k;u,d)$ a generating function for the generalized permutations of $[n]$ with exactly k ascents according to the statistics $\mathrm{nua}(\pi)$ and $\mathrm{nda}(\pi)$, that is

$$A(n,k;u,d) = \sum_{\pi\in\mathrm{GSA}_n} u^{\mathrm{nua}(\pi)} d^{\mathrm{nda}(\pi)}.$$

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We have

$$A(n,k;u,d) = \begin{cases} \neq 0 & \text{for } 0 \leq k < n \text{ for all } n \in \mathbb{N}, \\ = 0 & \text{otherwise.} \end{cases}$$

Recurrence relation

Theorem (H., 2020) (A336633)

The generating function $A(n,k;u,d)$ satisfies the following recurrence relation

$$A(n,k;u,d) = (k+1+(n-1)d)A(n-1,k;u,d) + (n-k+(n-1)u)A(n-1,k-1;u,d),$$

with initial condition $A(1,0;u,d) = 1$.

$$A(n;u,d) = \sum_{k=0}^{n-1} A(n,k;u,d)x^k$$

$A(1;u,d) = 1,$

$A(2;u,d) = (1+d) + (1+u)x,$

$A(3;u,d) = (2d^2+3d+1) + (4ud+4u+4d+1)x + (2u^2+3u+1)x^2$

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Reminder 1: degenerate Eulerian numbers of Carlitz (1979)

$$A_{n,k}(\lambda) = (k+1-(n-1)\lambda)A_{n-1,k}(\lambda) + (n-k+(n-1)\lambda)A_{n-1,k-1}(\lambda).$$

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Corollary (Connection to Carlitz's degenerate Eulerian numbers)

$$A(n,k;\lambda,-\lambda) = A_{n,k}(\lambda).$$

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$$A(n,k;u,d) = A(n,n-k-1;d,u), \quad 0 \leq k \leq n-1$$

Last notes...

Reminder 2: degenerate Eulerian numbers of Carlitz (1979)

$$\sum_{k=1}^{n} A_{n,k}(\lambda) = n!$$

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Theorem (H.,2020, A016777)

$$|GSA_n| = \prod_{k=1}^{n}(3k-2) = \sum_{j=0}^{n} |s(n, n-j)| \cdot 3^j,$$

where $s(n,k)$ are the Stirling numbers of the first kind.

Thank you for your attention