Combinatorics behind the degenerate Eulerian numbers

Orli Herscovici

School of Mathematics
Georgia Tech

Rutgers Experimental Mathematics Seminar
October 21th, 2021
Permutations: definitions

**Definition**

Permutation of a length $n$ is an ordering of the elements of the set $[n] = \{1, 2, \cdots, n\}$.

For any $\pi \in S_n$

$$\text{asc}(\pi) + \text{des}(\pi) = n - 1$$
Permutations: definitions

Definition

Permutation of a length $n$ is an ordering of the elements of the set $[n] = \{1, 2, \cdots, n\}$.

Definition

Let $\pi = \pi_1 \pi_2 \cdots \pi_n$ be a permutation. The index $k < n$ is

- an ascent of $\pi$ if $\pi_k < \pi_{k+1}$ ($\sim$ pair $\pi_k, \pi_{k+1}$ is a rise)
- a descent of $\pi$ if $\pi_k > \pi_{k+1}$ ($\sim$ pair $\pi_k, \pi_{k+1}$ is a fall)

For any $\pi \in S_n$

$$asc(\pi) + des(\pi) = n - 1$$
Permutations: definitions

Definition
Permutation of a length $n$ is an ordering of the elements of the set $[n] = \{1, 2, \cdots, n\}$.

Definition
Let $\pi = \pi_1 \pi_2 \cdots \pi_n$ be a permutation. The index $k < n$ is
- an ascent of $\pi$ if $\pi_k < \pi_{k+1}$ (\sim pair $\pi_k, \pi_{k+1}$ is a rise)
- a descent of $\pi$ if $\pi_k > \pi_{k+1}$ (\sim pair $\pi_k, \pi_{k+1}$ is a fall)

Example
Let $\pi = 31452$. 
Permutations: definitions

Definition
Permutation of a length \( n \) is an ordering of the elements of the set \([n] = \{1, 2, \cdots, n\}\).

Definition
Let \( \pi = \pi_1\pi_2\cdots\pi_n \) be a permutation. The index \( k < n \) is
- an ascent of \( \pi \) if \( \pi_k < \pi_{k+1} \) (\( \sim \) pair \( \pi_k, \pi_{k+1} \) is a rise)
- a descent of \( \pi \) if \( \pi_k > \pi_{k+1} \) (\( \sim \) pair \( \pi_k, \pi_{k+1} \) is a fall)

Example
Let \( \pi = 31452 \). Then
- \( asc(\pi) = \{2, 3\} \)
- \( des(\pi) = \{1, 4\} \)
**Definition**

Permutation of a length $n$ is an ordering of the elements of the set $[n] = \{1, 2, \cdots, n\}$.

**Definition**

Let $\pi = \pi_1 \pi_2 \cdots \pi_n$ be a permutation. The index $k < n$ is
- an ascent of $\pi$ if $\pi_k < \pi_{k+1}$ (\sim pair $\pi_k, \pi_{k+1}$ is a rise)
- a descent of $\pi$ if $\pi_k > \pi_{k+1}$ (\sim pair $\pi_k, \pi_{k+1}$ is a fall)

**Example**

Let $\pi = 31452$. Then

- $\text{asc}(\pi) = \{2, 3\}$
- $\text{des}(\pi) = \{1, 4\}$
- Ascending runs of $\pi$ are 3, 145
- Descending runs of $\pi$ are 31, 52
Permutations: definitions

**Definition**
Permutation of a length $n$ is an ordering of the elements of the set $[n] = \{1, 2, \cdots, n\}$.

**Definition**
Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation. The index $k < n$ is
- an ascent of $\pi$ if $\pi_k < \pi_{k+1}$ (pair $\pi_k, \pi_{k+1}$ is a rise)
- a descent of $\pi$ if $\pi_k > \pi_{k+1}$ (pair $\pi_k, \pi_{k+1}$ is a fall)

**Example**
Let $\pi = 31452$. Then
- $\text{asc}(\pi) = \{2, 3\}$
- $\text{des}(\pi) = \{1, 4\}$
- Ascending runs of $\pi$ are 3, 145
- Descending runs of $\pi$ are 31, 52

For any $\pi \in S_n$

$$\text{asc}(\pi) + \text{des}(\pi) = n - 1$$
Eulerian polynomials

Definition
Let us denote by $A(n, k)$ a number of permutations of length $n$ with exactly $k$ ascents. $A(n, k)$ are the Eulerian numbers.
Eulerian polynomials

Definition

Let us denote by $A(n, k)$ a number of permutations of length $n$ with exactly $k$ ascents. $A(n, k)$ are the Eulerian numbers.

For all integers $n \geq 0$, the $n$-th Eulerian polynomial is

$$A_n(t) = \sum_{k=0}^{n-1} A(n, k) t^k.$$
Eulerian polynomials

Definition

Let us denote by $A(n, k)$ a number of permutations of length $n$ with exactly $k$ ascents. $A(n, k)$ are the Eulerian numbers.

For all integers $n \geq 0$, the $n$-th Eulerian polynomial is

$$A_n(t) = \sum_{k=0}^{n-1} A(n, k) t^k.$$ 

Generating functions

$$\frac{t - 1}{t - e^{u(t-1)}} = \sum_{n \geq 0} \sum_{k \geq 0} A(n, k) t^k \frac{u^n}{n!},$$

$$\frac{A_n(t)}{(1 - t)^{n+1}} = \sum_{j \geq 0} (j + 1)^n t^j.$$
Different notations for the Eulerian polynomial

Today:

\[
\frac{1 - t}{1 - te^{u(1-t)}} = \sum_{n \geq 0} \sum_{k \geq 0} A(n, k) t^k \frac{u^n}{n!},
\]

Euler:

\[
\frac{t - 1}{t - e^{u(t-1)}} = \sum_{n \geq 0} \sum_{k \geq 0} A(n, k) t^k \frac{u^n}{n!},
\]
Different notations for the Eulerian polynomial

Today:

\[
\frac{1 - t}{1 - te^{u(1-t)}} = \sum_{n \geq 0} \sum_{k \geq 0} A(n, k) t^k u^n / n!,
\]

\[
A_0(t) = 1,
A_1(t) = t,
A_2(t) = t^2 + t,
A_3(t) = t^3 + 4t^2 + t,
A_4(t) = t^4 + 11t^3 + 11t^2 + t,
\]

Euler:

\[
\frac{t - 1}{t - e^{u(t-1)}} = \sum_{n \geq 0} \sum_{k \geq 0} A(n, k) t^k u^n / n!,
\]

\[
A_0(t) = 1,
A_1(t) = 1,
A_2(t) = t + 1,
A_3(t) = t^2 + 4t + 1,
A_4(t) = t^3 + 11t^2 + 11t + 1,
\]
Different notations for the Eulerian polynomial

Today:

\[ \frac{1 - t}{1 - te^{u(1-t)}} = \sum_{n \geq 0} \sum_{k \geq 0} A(n, k) t^k \frac{u^n}{n!}, \]

\[ A_0(t) = 1, \]
\[ A_1(t) = t, \]
\[ A_2(t) = t^2 + t, \]
\[ A_3(t) = t^3 + 4t^2 + t, \]
\[ A_4(t) = t^4 + 11t^3 + 11t^2 + t, \]

equation of ascending runs

Euler:

\[ \frac{t - 1}{t - e^{u(t-1)}} = \sum_{n \geq 0} \sum_{k \geq 0} A(n, k) t^k \frac{u^n}{n!}, \]

\[ A_0(t) = 1, \]
\[ A_1(t) = 1, \]
\[ A_2(t) = t + 1, \]
\[ A_3(t) = t^2 + 4t + 1, \]
\[ A_4(t) = t^3 + 11t^2 + 11t + 1, \]

equation of ascents

Orli Herscovici (Georgia Tech)
Eulerian numbers
Rutgers 4/25
Generalizations: $q$-calculus

$q$-integers:

\[ [n]_q := \frac{1 - q^n}{1 - q} = \sum_{j=0}^{n-1} q^j \]

Carlitz $q$-Eulerian pol. (1954)

\[ \frac{A_n(t, q)}{(t; q)_{n+1}} = \sum_{j \geq 0} t^j ([j + 1]_q)^n \]

\begin{align*}
A_0(t, q) &= 1, & A_1(t, q) &= 1, & A_2(t, q) &= qt + 1, \\
A_3(t, q) &= q^3 t^2 + 2q(q + 1)t + 1, \\
A_4(t, q) &= q^6 t^3 + (3q^2 + 5q + 3)q^3 t^2 + (3q^2 + 5q + 3)qt + 1,
\end{align*}
Generalizations: $q$-calculus

$q$-integers:

$$[n]_q := \frac{1 - q^n}{1 - q} = \sum_{j=0}^{n-1} q^j$$

$q$-ascending factorial:

$$(x; q)_k := \begin{cases} 1, & k = 0, \\ \prod_{j=0}^{k-1} (1 - q^j x), & k \geq 1. \end{cases}$$

Eulerian polynomials

$$A_n(t, q) = \sum_{j \geq 0} t^j (j + 1) [j + 1]_q^n \cdot (1 - q^n t) = \prod_{j=0}^{n-1} (1 - q^j x),$$

$$A_0(t, q) = 1, \quad A_1(t, q) = 1, \quad A_2(t, q) = qt + 1,$$

$$A_3(t, q) = q^3 t^2 + 2q(q + 1)t + 1,$$

$$A_4(t, q) = q^6 t^3 + (3q^2 + 5q + 3)q^3 t^2 + (3q^2 + 5q + 3)qt + 1,$$

Orli Herscovici (Georgia Tech)

Eulerian numbers

Rutgers 5 / 25
Generalizations: *q*-calculus

**q-integers:**

\[
[n]_q := \frac{1 - q^n}{1 - q} = \sum_{j=0}^{n-1} q^j
\]

\[
\lim_{q \to 1} [j + 1]_q = j + 1,
\]

**q-ascending factorial:**

\[
(x; q)_k := \begin{cases} 
1, & k = 0, \\
\prod_{j=0}^{k-1} (1 - q^j x), & k \geq 1.
\end{cases}
\]

\[
\lim_{q \to 1} (x; q)_{n+1} = (1 - x)^{n+1}.
\]
Generalizations: $q$-calculus

$q$-integers:

$$[n]_q := \frac{1 - q^n}{1 - q} = \sum_{j=0}^{n-1} q^j$$

$q$-ascending factorial:

$$(x; q)_k := \begin{cases} 1, & k = 0, \\ \prod_{j=0}^{k-1} (1 - q^j x), & k \geq 1. \end{cases}$$

$$\lim_{q \to 1} [j + 1]_q = j + 1,$$

$$\lim_{q \to 1} (x; q)_{n+1} = (1 - x)^{n+1}.$$
Generalizations: $q$-calculus

$q$-integers:

$$[n]_q := \frac{1 - q^n}{1 - q} = \sum_{j=0}^{n-1} q^j$$

$$\lim_{q \to 1} [j + 1]_q = j + 1,$$

$q$-ascending factorial:

$$(x; q)_k := \begin{cases} 1, & k = 0, \\ \prod_{j=0}^{k-1} (1 - q^j x), & k \geq 1. \end{cases}$$

$$\lim_{q \to 1} (x; q)_{n+1} = (1 - x)^{n+1}.$$

Eulerian polynomials

$$\frac{A_n(t)}{(1 - t)^{n+1}} = \sum_{j \geq 0} t^j (j + 1)^n$$

Carlitz $q$-Eulerian pol. (1954)

$$\frac{A_n(t, q)}{(t; q)_{n+1}} = \sum_{j \geq 0} t^j ([j + 1]_q)^n$$
Generalizations: \( q \)-calculus

\textbf{\( q \)-integers:}

\[ [n]_q := \frac{1 - q^n}{1 - q} = \sum_{j=0}^{n-1} q^j \]

\textbf{\( q \)-ascending factorial:}

\[ (x; q)_k := \begin{cases} 1, & k = 0, \\ \prod_{j=0}^{k-1} (1 - q^j x), & k \geq 1. \end{cases} \]

\[ \lim_{q \to 1} [j + 1]_q = j + 1, \]

\[ \lim_{q \to 1} (x; q)_{n+1} = (1 - x)^{n+1}. \]

\textbf{Eulerian polynomials}

\[ \frac{A_n(t)}{(1 - t)^{n+1}} = \sum_{j \geq 0} t^j (j + 1)^n \]

\textbf{Carlitz \( q \)-Eulerian pol. (1954)}

\[ \frac{A_n(t, q)}{(t; q)_{n+1}} = \sum_{j \geq 0} t^j ([j + 1]_q)^n \]

\[ A_0(t, q) = 1, \quad A_1(t, q) = 1, \quad A_2(t, q) = qt + 1, \]

\[ A_3(t, q) = q^3 t^2 + 2q(q + 1)t + 1, \]

\[ A_4(t, q) = q^6 t^3 + (3q^2 + 5q + 3)q^3 t^2 + (3q^2 + 5q + 3)qt + 1, \]
\(q\)-Eulerian polynomials of Carlitz

Carlitz \(q\)-Eulerian pol. (1954)

\[
\frac{A_n(t, q)}{(t; q)_{n+1}} = \sum_{j \geq 0} \begin{pmatrix} j + 1 \end{pmatrix}_q^n t^j (\begin{pmatrix} j + 1 \end{pmatrix}_q^n).
\]

\[A_4(t, q) = q^6 t^3 + (3q^2 + 5q + 3)q^3 t^2 + (3q^2 + 5q + 3)qt + 1,\]

\(q\)-Eulerian polynomials (Carlitz, 1974)

\[A_n(t, q) = \sum_{\pi \in S_n} t^{\text{des}(\pi)} q^{\text{maj}(\pi)}, \text{ where } \text{maj}(\pi) := \sum_{\pi_i > \pi_{i+1}} i.\]
$q$-Eulerian polynomials of Carlitz

Carlitz $q$-Eulerian pol. (1954)

\[
\frac{A_n(t, q)}{(t; q)_{n+1}} = \sum_{j \geq 0} t^j ([j + 1]_q)^n
\]

\[
A_4(t, q) = q^6 t^3 + (3q^2 + 5q + 3)q^3 t^2 + (3q^2 + 5q + 3)qt + 1,
\]

Example

\[
\pi = 4321
\]

- $\text{des}(\pi) = \{1, 2\}$
- $1 + 2 = 3$.

$q$-Eulerian polynomials (Carlitz, 1974)

\[
A_n(t, q) = \sum_{\pi \in S_n} t^{\text{des}(\pi)} q^{\text{maj}(\pi)}, \quad \text{where } \text{maj}(\pi) := \sum_{\pi_i > \pi_{i+1}} i
\]
 Carlitz $q$-Eulerian pol. (1954)

\[
\frac{A_n(t, q)}{(t; q)_{n+1}} = \sum_{j \geq 0} t^j ([j + 1]_q)^n
\]

\[
A_4(t, q) = q^6 t^3 + (3q^2 + 5q + 3)q^3 t^2 + (3q^2 + 5q + 3)qt + 1,
\]

**Example**

\[
\pi = 4321
\]

\[
\text{des}(\pi) = \{1, 2, 3\}
\]

\[
1 + 2 = 3.
\]
q-Eulerian polynomials of Carlitz

Carlitz q-Eulerian pol. (1954)

\[
\frac{A_n(t, q)}{(t; q)_{n+1}} = \sum_{j \geq 0} t^j ([j + 1]_q)^n
\]

\[
A_4(t, q) = q^6 t^3 + (3q^2 + 5q + 3)q^3 t^2 + (3q^2 + 5q + 3)qt + 1,
\]

Example

\(\pi = 4321\)

- \(\text{des}(\pi) = \{1, 2, 3\}\)
- \(1 + 2 + 3 = 6.\)
\( q \)-Eulerian polynomials of Carlitz

**Carlitz \( q \)-Eulerian pol. (1954)**

\[
\frac{A_n(t, q)}{(t; q)_{n+1}} = \sum_{j \geq 0} t^j ([j+1]_q)^n
\]

\[
A_4(t, q) = q^6 t^3 + (3q^2 + 5q + 3)q^3 t^2 + (3q^2 + 5q + 3)qt + 1,
\]

**Example**

\( \pi = 4321 \)

- \( des(\pi) = \{1, 2, 3\} \)
- \( 1 + 2 + 3 = 6. \)

\( \pi = 4312, 4213, 3214 \)
$q$-Eulerian polynomials of Carlitz

Carlitz $q$-Eulerian pol. (1954)

$$\frac{A_n(t, q)}{(t; q)_{n+1}} = \sum_{j \geq 0} t^j ([j + 1]_q)_n$$

$$A_4(t, q) = q^6 t^3 + (3q^2 + 5q + 3)q^3 t^2 + (3q^2 + 5q + 3)qt + 1,$$

Example

\(\pi = 4321\)
- \(\text{des}(\pi) = \{1, 2, 3\}\)
- \(1 + 2 + 3 = 6.\)

\(\pi = 4312, 4213, 3214\)
- \(\text{des}(\pi) = \{1, 2\}\)
Carlitz $q$-Eulerian polynomials (1954)

$$A_n(t, q) = \frac{A_n(t, q)}{(t; q)_{n+1}} = \sum_{j \geq 0} t^j ([j + 1]_q)^n$$

$$A_4(t, q) = q^6 t^3 + (3q^2 + 5q + 3)q^3 t^2 + (3q^2 + 5q + 3)qt + 1,$$

Example

$\pi = 4321$
- $\text{des}(\pi) = \{1, 2, 3\}$
- $1 + 2 + 3 = 6.$

$\pi = 4312, 4213, 3214$
- $\text{des}(\pi) = \{1, 2\}$
- $1 + 2 = 3.$
$$A_n(t, q) = \sum_{j \geq 0} t^j ([j + 1]_q)^n$$

$$A_4(t, q) = q^6 t^3 + (3q^2 + 5q + 3)q^3 t^2 + (3q^2 + 5q + 3)qt + 1,$$

Example

$$\pi = 4321$$
- $$\text{des}(\pi) = \{1, 2, 3\}$$
- $$1 + 2 + 3 = 6.$$ 

$$\pi = 4312, 4213, 3214$$
- $$\text{des}(\pi) = \{1, 2\}$$
- $$1 + 2 = 3.$$ 

$$q$$-Eulerian polynomials (Carlitz, 1974)

$$A_n(t, q) = \sum_{\pi \in S_n} t^{\text{des}(\pi)} q^{\text{maj}(\pi)},$$

where $$\text{maj}(\pi) := \sum_{\pi_j > \pi_{j+1}} i$$
Generalized Eulerian numbers [Carlitz, Scoville 1974]

\( A_{n,k} \) - \# of ascending runs

\[
\frac{1 - y}{1 - ye^{x(1-y)}} = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{k=1}^{n} A_{n,k} y^k
\]

Symmetry relation:

\[
A_{n,k} = A_{n,n-k+1}
\]
Generalized Eulerian numbers [Carlitz, Scoville 1974]

\( A_{n,k} \) - \# of ascending runs

\[
\frac{1 - y}{1 - ye^{x(1-y)}} = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{k=1}^{n} A_{n,k} y^k
\]

Symmetry relation:

\( A_{n,k} = A_{n,n-k+1} \)

\[
A(r, s) = A_{r+s+1,s+1} = A_{r+s+1,r+1} = A(s, r).
\]
Generalized Eulerian numbers [Carlitz, Scoville 1974]

\[ A_{n,k} \text{ - # of ascending runs} \]

\[
\frac{1 - y}{1 - ye^{x(1-y)}} = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{k=1}^{n} A_{n,k} y^k
\]

Symmetry relation:

\[ A_{n,k} = A_{n,n-k+1} \]

\[ A(r, s) = A_{r+s+1, s+1} = A_{r+s+1, r+1} = A(s, r). \]

\[
\sum_{r,s=0}^{\infty} A(r, s) \frac{x^r y^s}{(r + s + 1)!} = \frac{e^x - e^y}{xe^y - ye^x} = F(x, y)
\]
Generalized Eulerian numbers [Carlitz, Scoville 1974]

\(A_{n,k} - \) \# of ascending runs

\[
\frac{1 - y}{1 - ye^{x(1-y)}} = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{k=1}^{n} A_{n,k} y^k
\]

Symmetry relation:

\[A_{n,k} = A_{n,n-k+1}\]

\[A(r, s) = A_{r+s+1,s+1} = A_{r+s+1,r+1} = A(s, r).\]

\[
\sum_{r,s=0}^{\infty} A(r, s) \frac{x^r y^s}{(r + s + 1)!} = \frac{e^x - e^y}{xe^y - ye^x} = F(x, y)
\]

\[
\sum_{r,s=0}^{\infty} A(r, s) \frac{x^r y^s}{(r + s)!} = (1 + xF(x, y))(1 + yF(x, y))
\]
Generalized Eulerian numbers [Carlitz, Scoville 1974]

\[ \sum_{r,s=0}^{\infty} A(r, s) \frac{x^r y^s}{(r + s)!} = (1 + xF(x, y))(1 + yF(x, y)) \]

**Properties:**

\[ A(r, s|\alpha, \beta) = A(s, r|\beta, \alpha) \]

\[ A(r, s|\alpha, \beta) = (r + \beta)A(r, s-1|\alpha, \beta) + (s + \alpha)A(r-1, s|\alpha, \beta) \]

NOTE: Polynomial in \(\alpha, \beta\) with positive integer coefficents.
Generalized Eulerian numbers [Carlitz, Scoville 1974]

\[
\sum_{r,s=0}^{\infty} A(r, s) \frac{x^r y^s}{(r + s)!} = (1 + xF(x, y))(1 + yF(x, y))
\]

\[
\sum_{r,s=0}^{\infty} A(r, s|\alpha, \beta) \frac{x^r y^s}{(r + s)!} = (1 + xF(x, y))^\alpha(1 + yF(x, y))^\beta
\]
Generalized Eulerian numbers [Carlitz, Scoville 1974]

\[
\sum_{r,s=0}^{\infty} A(r, s) \frac{x^r y^s}{(r+s)!} = (1 + xF(x, y))(1 + yF(x, y))
\]

\[
\sum_{r,s=0}^{\infty} A(r, s|\alpha, \beta) \frac{x^r y^s}{(r+s)!} = (1 + xF(x, y))^{\alpha}(1 + yF(x, y))^{\beta}
\]

Properties:

\[
A(r, s|1, 1) = A(r, s),
\]

\[
A(r, s|\alpha, \beta) = A(s, r|\beta, \alpha)
\]
Generalized Eulerian numbers [Carlitz, Scoville 1974]

\[
\sum_{r,s=0}^{\infty} A(r, s) \frac{x^r y^s}{(r + s)!} = (1 + xF(x, y))(1 + yF(x, y))
\]

\[
\sum_{r,s=0}^{\infty} A(r, s|\alpha, \beta) \frac{x^r y^s}{(r + s)!} = (1 + xF(x, y))^{\alpha}(1 + yF(x, y))^{\beta}
\]

Properties:

\[
A(r, s|1, 1) = A(r, s),
\]
\[
A(r, s|\alpha, \beta) = A(s, r|\beta, \alpha)
\]

\[
A(r, s|\alpha, \beta) = (r + \beta)A(r, s - 1|\alpha, \beta) + (s + \alpha)A(r - 1, s|\alpha, \beta)
\]
Generalized Eulerian numbers [Carlitz, Scoville 1974]

\[
\sum_{r,s=0}^{\infty} A(r, s) \frac{x^r y^s}{(r + s)!} = (1 + xF(x, y))(1 + yF(x, y))
\]

\[
\sum_{r,s=0}^{\infty} A(r, s|\alpha, \beta) \frac{x^r y^s}{(r + s)!} = (1 + xF(x, y))^\alpha(1 + yF(x, y))^\beta
\]

Properties:

\[A(r, s|1, 1) = A(r, s),\]
\[A(r, s|\alpha, \beta) = A(s, r|\beta, \alpha)\]

\[A(r, s|\alpha, \beta) = (r + \beta)A(r, s - 1|\alpha, \beta) + (s + \alpha)A(r - 1, s|\alpha, \beta)\]
Combinatorial applications of $A(r, s|\alpha, \beta)$

Generalization of

- rises
- falls
- maxima
Combinatorial applications of $A(r, s|\alpha, \beta)$

Generalization of

- rises
- falls
- maxima

Two kinds of “virtual” elements: 0 and 0′

$\alpha$  # of symbols 0
$\beta$  # of symbols 0′
Combinatorial applications of $A(r, s|\alpha, \beta)$

Generalization of

- rises
- falls
- maxima

Two kinds of “virtual” elements: 0 and 0’

$\alpha$ # of symbols 0
$\beta$ # of symbols 0’

Construction: initial step ($n = 1$)

$\underbrace{0 \ldots 0}_{\alpha} 1 \underbrace{0’ \ldots 0’}_{\beta}$
Combinatorial applications of $A(r, s|\alpha, \beta)$

Generalization of

**rises** a pair of consecutive elements $a, b$ with $a < b$, where $a$ may be 0

**falls**

**maxima**

Two kinds of “virtual” elements: 0 and 0’

$\alpha$ # of symbols 0

$\beta$ # of symbols 0’

Construction: initial step ($n = 1$)

$$0 \ldots 0 1 0’ \ldots 0’$$

\[\alpha \quad \beta\]
Combinatorial applications of $A(r, s|\alpha, \beta)$

Generalization of

- **rises** a pair of consecutive elements $a, b$ with $a < b$, where $a$ may be 0
- **falls** a pair of consecutive elements $a, b$ with $a > b$, where $b$ may be 0’

**maxima**

Two kinds of “virtual” elements: 0 and 0’

- $\alpha$ # of symbols 0
- $\beta$ # of symbols 0’

**Construction: initial step ($n = 1$)**

$$0 \ldots 0 \underbrace{1 \, 0^\prime \ldots 0^\prime}_{\alpha} \underbrace{0^\prime \ldots 0^\prime}_{\beta}$$
Combinatorial applications of $A(r, s|\alpha, \beta)$

Generalization of

**rises** a pair of consecutive elements $a, b$ with $a < b$, where $a$ may be 0

**falls** a pair of consecutive elements $a, b$ with $a > b$, where $b$ may be 0’

**maxima** the element $b$ is a maximum if $a, b, c$ are consecutive and $a, b$ is a rise while $b, c$ is a fall.

Two kinds of “virtual” elements: 0 and 0’

$\alpha \#$ of symbols 0

$\beta \#$ of symbols 0’

Construction: initial step ($n = 1$)

$\underbrace{0 \ldots 0}_{\alpha} \ 1 \underbrace{0' \ldots 0'}_{\beta}$
Construction: requirement

Symbols 2, 3, ..., n can be inserted in all possible places w.r.t the requirements

1. there is at least one 0 on the extreme left
2. there is at least one 0′ on the extreme right

Example (Carlitz, Scoville 1974)

0 ...

α

1 0

′ ...

β

1 rise
1 fall
1 maximum

α = 2
β = 3

# of rises r = 4
# of falls s = 3
# of maxima k = 1
Construction: requirement

Symbols 2, 3, . . . , n can be inserted in all possible places w.r.t the requirements

1. there is at least one 0 on the extreme left
2. there is at least one 0′ on the extreme right

Example (Carlitz, Scoville 1974)

\[ \underbrace{0 \ldots 0}_\alpha 1 \underbrace{0' \ldots 0'}_\beta \]

- 1 rise
- 1 fall
- 1 maximum

- # of rises \( r = 4 \)
- # of falls \( s = 3 \)
- # of maxima \( k = 1 \)
Combinatorial applications of $A(r, s|\alpha, \beta)$

**Construction: requirement**

Symbols 2, 3, ..., $n$ can be inserted in all possible places w.r.t the requirements

1. there is at least one 0 on the extreme left
2. there is at least one 0' on the extreme right

**Example (Carlitz, Scoville 1974)**

$$0 \ldots 0 \underbrace{1 \ 0'} \ldots \underbrace{0'}_{\alpha \ \beta}$$

- 1 rise
- 1 fall
- 1 maximum

02301540'0'60'

- # of falls $s = 3$
- # of maxima $k = 1$
Combinatorial applications of $A(r, s | \alpha, \beta)$

**Construction: requirement**

Symbols $2, 3, \ldots, n$ can be inserted in all possible places w.r.t the requirements

1. there is at least one 0 on the extreme left
2. there is at least one 0′ on the extreme right

**Example (Carlitz, Scoville 1974)**

$$\underbrace{0 \ldots 0}_{\alpha} 1 \underbrace{0' \ldots 0'}_{\beta}$$

02301540’0’60’

- 1 rise
- 1 fall
- 1 maximum

$\alpha = 2$

$\#$ of maxima $k = 1$
Combinatorial applications of $A(r, s \mid \alpha, \beta)$

**Construction: requirement**

Symbols $2, 3, \ldots, n$ can be inserted in all possible places w.r.t the requirements

1. there is at least one 0 on the extreme left
2. there is at least one 0' on the extreme right

**Example (Carlitz, Scoville 1974)**

\[
\underbrace{0 \ldots 0}_\alpha \underbrace{1 0' \ldots 0'}_\beta \quad 02301540'0'60'
\]

- 1 rise
- 1 fall
- 1 maximum

\[\alpha = 2, \beta = 3\]
Combinatorial applications of $A(r, s|\alpha, \beta)$

**Construction: requirement**

Symbols 2, 3, ..., $n$ can be inserted in all possible places w.r.t the requirements

1. there is at least one 0 on the extreme left
2. there is at least one 0' on the extreme right

**Example (Carlitz, Scoville 1974)**

$$0 \ldots 0 1 0' \ldots 0'$$

- 1 rise
- 1 fall
- 1 maximum

$$02301540'0'60'$$

- $\alpha = 2$
- $\beta = 3$
- # of rises $r = 4$
Combinatorial applications of $A(r, s|\alpha, \beta)$

Construction: requirement
Symbols 2, 3, . . . , n can be inserted in all possible places w.r.t the requirements
1. there is at least one 0 on the extreme left
2. there is at least one 0′ on the extreme right

Example (Carlitz, Scoville 1974)

\[0\ldots0\underbrace{10\ldots0}_\alpha\underbrace{0\ldots0}_\beta\]

- 1 rise
- 1 fall
- 1 maximum

\[02301540'0'60'\]
- $\alpha = 2$
- $\beta = 3$
- # of rises $r = 4$
- # of falls $s = 3$
Construction: requirement

Symbols $2, 3, \ldots, n$ can be inserted in all possible places w.r.t the requirements

1. there is at least one 0 on the extreme left
2. there is at least one 0’ on the extreme right

Example (Carlitz, Scoville 1974)

\[
\underbrace{0 \ldots 0}_{\alpha} \underbrace{1 0' \ldots 0'}_{\beta}
\]

- 1 rise
- 1 fall
- 1 maximum

\[
02301540'0'60'
\]

\[
\begin{align*}
\alpha &= 2 \\
\beta &= 3 \\
\# \text{ of rises } r &= 4 \\
\# \text{ of falls } s &= 3 \\
\# \text{ of maxima } k &= 1
\end{align*}
\]
Combinatorial applications of $A(r, s|\alpha, \beta)$

Let $P(r, s|\alpha, \beta)$ be the number of $(\alpha, \beta)$-sequences with $r$ rises and $s$ falls.
Combinatorial applications of $A(r, s | \alpha, \beta)$

Let $P(r, s | \alpha, \beta)$ be the number of $(\alpha, \beta)$-sequences with $r$ rises and $s$ falls

the number of real elements $n = r + s - 1$
Combinatorial applications of $A(r, s|\alpha, \beta)$

Let $P(r, s|\alpha, \beta)$ be the number of $(\alpha, \beta)$-sequences with $r$ rises and $s$ falls.

The number of real elements $n = r + s - 1$

$$P(r, s|\alpha, \beta) = (r + \beta - 1)P(r, s - 1|\alpha, \beta) + (s + \alpha - 1)P(r - 1, s|\alpha, \beta)$$

$$P(r, 1|\alpha, \beta) = \alpha^{r-1}, \quad P(1, s|\alpha, \beta) = \beta^{s-1}$$
Combinatorial applications of $A(r, s|\alpha, \beta)$

Let $P(r, s|\alpha, \beta)$ be the number of $(\alpha, \beta)$-sequences with $r$ rises and $s$ falls

the number of real elements $n = r + s - 1$

\[ P(r, s|\alpha, \beta) = (r + \beta - 1)P(r, s - 1|\alpha, \beta) + (s + \alpha - 1)P(r - 1, s|\alpha, \beta) \]

\[ P(r, 1|\alpha, \beta) = \alpha^{r-1}, \quad P(1, s|\alpha, \beta) = \beta^{s-1} \]

\[ P(r + 1, s + 1|\alpha, \beta) = A(r, s|\alpha, \beta) \]
Carlitz degenerate Eulerian polynomials

Degenerate exponential function of Carlitz, 1956

\[(1 + \lambda x)^\mu, \quad \lambda \mu = 1.\]
Carlitz degenerate Eulerian polynomials

Degenerate exponential function of Carlitz, 1956

\[(1 + \lambda x)^\mu, \quad \lambda \mu = 1.\]

Degenerate Eulerian numbers (Carlitz, 1979)

\[\frac{1 - t}{1 - t(1 + \lambda(u - tu))^\mu} = 1 + \sum_{n=1}^{\infty} \frac{u^n}{n!} \sum_{k=1}^{n} A_{n,k}(\lambda) t^k.\]
Carlitz degenerate Eulerian polynomials

Degenerate exponential function of Carlitz, 1956

\[(1 + \lambda x)^\mu, \quad \lambda \mu = 1.\]

Degenerate Eulerian numbers (Carlitz, 1979)

\[
\frac{1 - t}{1 - t(1 + \lambda(u - tu))^\mu} = 1 + \sum_{n=1}^{\infty} \frac{u^n}{n!} \sum_{k=1}^{n} A_{n,k}(\lambda) t^k.
\]

\[A_0 = 1,\]
\[A_1 = t,\]
\[A_2 = (\lambda + 1)t^2 + (-\lambda + 1)t,\]
\[A_3 = (2\lambda^2 + 3\lambda + 1)t^3 + (-4\lambda^2 + 4)t^2 + (2\lambda^2 - 3\lambda + 1)t, \ldots\]
### Carlitz degenerate Eulerian polynomials

#### Degenerate exponential function of Carlitz, 1956

\[(1 + \lambda x)^\mu, \quad \lambda \mu = 1.\]

#### Degenerate Eulerian numbers (Carlitz, 1979)

\[
\frac{1 - t}{1 - t(1 + \lambda(u - tu))^{\mu}} = 1 + \sum_{n=1}^{\infty} \frac{u^n}{n!} \sum_{k=1}^{n} A_{n,k}(\lambda) t^k.
\]

\[
A_0 = 1,
\]
\[
A_1 = t,
\]
\[
A_2 = (\lambda + 1)t^2 + (-\lambda + 1)t,
\]
\[
A_3 = (2\lambda^2 + 3\lambda + 1)t^3 + (-4\lambda^2 + 4)t^2 + (2\lambda^2 - 3\lambda + 1)t, \ldots
\]

**NOTE:**

\[
\sum_{k=1}^{n} A_{n,k}(\lambda) = n!
\]
Carlitz degenerate Eulerian numbers

Symmetry:

\[ A_{n,n-k+1}(\lambda) = A_{n,k}(-\lambda) \]
Carlitz degenerate Eulerian numbers

Symmetry:

\[ A_{n,n-k+1}(\lambda) = A_{n,k}(-\lambda) \]

Recurrence:

\[ A_{n,k}(\lambda) = (k + 1 - (n - 1)\lambda)A_{n-1,k}(\lambda) \]
\[ + (n - k + (n - 1)\lambda)A_{n-1,k-1}(\lambda), \]

with an initial condition \( A_{0,0}(\lambda) = 1. \)
Carlitz degenerate Eulerian numbers

Symmetry:

\[ A_{n,n-k+1}(\lambda) = A_{n,k}(-\lambda) \]

Recurrence:

\[
A_{n,k}(\lambda) = (k + 1 - (n - 1)\lambda)A_{n-1,k}(\lambda) \\
+ (n - k + (n - 1)\lambda)A_{n-1,k-1}(\lambda),
\]

with an initial condition \( A_{0,0}(\lambda) = 1. \)

The numbers \( S(n,k), S_1(n,k), B_n, A_{n,k} \) have various arithmetic and combinatorial properties. We do not consider any such questions for the quantities discussed in the present paper but hope to do so in a later communication.
Now we rotate each tile 180°:
The permutation is not changed. Ascents and descents are not changed.

Let’s change it! How?
Generalization of permutations

Now we rotate each tile 180°:

The permutation is not changed. Ascents and descents are not changed.

Let’s change it! How?
Generalization of permutations

Let us illustrate a permutation \( \pi = 31452 \) in a domino tiles style:

![Domino tiles style illustration](image)

Let's change it! How?

Orli Herscovici (Georgia Tech)
Generalization of permutations

Let us illustrate a permutation $\pi = 31452$ in a domino tiles style:

Now we rotate each tile $180^\circ$:

The permutation is not changed. Ascents and descents are not changed.

Let's change it! How?
Let us illustrate a permutation $\pi = 31452$ in a domino tiles style:

Now we rotate each tile $180^\circ$:

The permutation is not changed. Ascents and descents are not changed.
Generalization of permutations

Let us illustrate a permutation $\pi = 31452$ in a domino tiles style:

Now we rotate each tile $180^\circ$:

The permutation is not changed. Ascents and descents are not changed.

Let’s change it! How?

Orli Herscovici (Georgia Tech)
Eulerian numbers
Rutgers 14/25
Generalized permutations

More tiles:

[Diagram of two tiles arranged in a 2x2 grid, with one tile pointing north and another to the right]
Generalized permutations

More tiles:

\[
\begin{array}{c}
\uparrow \\
. \\
\downarrow \\
\end{array}
\rightarrow
after rotation by 180°

\begin{array}{c}
. \\
\downarrow
\end{array}
\]

For each \( n \in \mathbb{N} \)

\[
\begin{array}{c}
n \\
\end{array}
\rightarrow
\begin{array}{c}
n \\
\uparrow \\
\downarrow
\end{array}
\]
Generalized permutations

More tiles:

For each $n \in \mathbb{N}$

Number of permutations:

$$n! \implies (3n)!!! \quad (= 3^n n!)$$

where $(3n)!!! = \prod_{i=1}^{n}(3i)$. **Triple permutations**
Example (Ascents in classical permutation)

Let $\pi = 31452$. Then

- $\text{asc}(\pi) = \{2, 3\}$
- $\text{des}(\pi) = \{1, 4\}$

Question:

How can we define ascents in triple permutations?

Basic rule:

Upward arrow tile adds an ascent!

We insert an upward arrow tile $n \uparrow$ and a downward arrow tile $n \downarrow$ on the left from permutation tiles.
Example (Ascents in classical permutation)

Let $\pi = 31452$. Then

- $\text{asc}(\pi) = \{2, 3\}$
- $\text{des}(\pi) = \{1, 4\}$

Question:

How can we define ascents in triple permutations?
Example (Ascents in classical permutation)

Let $\pi = 31452$. Then

- $\text{asc}(\pi) = \{2, 3\}$
- $\text{des}(\pi) = \{1, 4\}$

Question:

How can we define ascents in triple permutations?

Basic rule:

Upward arrow tile adds an ascent!
Example (Ascents in classical permutation)

Let $\pi = 31452$. Then

- $\text{asc}(\pi) = \{2, 3\}$
- $\text{des}(\pi) = \{1, 4\}$

Question:

How can we define ascents in triple permutations?

Basic rule:

Upward arrow tile adds an ascent!

We insert an upward arrow tile $\uparrow$ and a downward arrow tile $\downarrow$ on the left from permutation tiles.
Generalized permutations: construction

Recursive construction of $\pi' = \pi_1' \cdots \pi_n'$ from $\pi = \pi_1 \cdots \pi_{n-1}$

4. $\pi_1 \cdots \pi_i \uparrow \pi_{i+1} \cdots \pi_{n-1}$, where $i \in \text{des}(\pi)$, then $|\text{asc}(\pi')| = |\text{asc}(\pi)| + 1$
   $\text{asc}(\pi') = \{k < i \mid k \in \text{asc}(\pi)\} \cup \{i\} \cup \{k + 1 \mid k \in \text{asc}(\pi), k \geq i\}$

5. $\pi_1 \cdots \pi_{i-1} \downarrow \pi_i \cdots \pi_{n-1}$
   then $|\text{asc}(\pi')| = |\text{asc}(\pi)| + 1$
   $\text{asc}(\pi') = \{k < i \mid k \in \text{asc}(\pi)\} \cup \{i\} \cup \{k + 1 \mid k \in \text{asc}(\pi), k \geq i\}$

6. $\downarrow \pi_1 \cdots \pi_{n-1}$ does not change the number of existing ascents of permutations.
Generalized permutations: construction

Recursive construction of $\pi' = \pi'_1 \cdots \pi'_n$ from $\pi = \pi_1 \cdots \pi_{n-1}$

1. $\pi_1 \cdots \pi_{n-1} \Rightarrow |asc(\pi')| = |asc(\pi)|$, $\text{asc}(\pi') = \{ k \mid k - 1 \in \text{asc}(\pi) \}$

5. $\pi_1 \cdots \pi_{i-1} \overset{n \uparrow}{\rightarrow} \pi_i \cdots \pi_{n-1}$
   then $|\text{asc}(\pi')| = |\text{asc}(\pi)| + 1,$
   $\text{asc}(\pi') = \{ k < i \mid k \in \text{asc}(\pi) \} \cup \{i\} \cup \{ k + 1 \mid k \in \text{asc}(\pi), \; k \geq i \}$

6. $\downarrow{n}$ does not change a number of existing ascents of permutations.
Generalized permutations: construction

Recursive construction of $\pi' = \pi_1' \cdots \pi_n'$ from $\pi = \pi_1 \cdots \pi_{n-1}$

1. $\square^n \pi_1 \cdots \pi_{n-1} \Rightarrow |asc(\pi')| = |asc(\pi)|, \quad asc(\pi') = \{k \mid k - 1 \in asc(\pi)\}$

2. $\pi_1 \cdots \pi_{n-1} \quad \square^n \Rightarrow |asc(\pi')| = |asc(\pi)| + 1, \quad asc(\pi') = asc(\pi) \cup \{n - 1\}$

6. $\downarrow^n$ does not change a number of existing ascents of permutations.
**Generalized permutations: construction**

**Recursive construction of** \( \pi' = \pi'_1 \cdots \pi'_n \) **from** \( \pi = \pi_1 \cdots \pi_{n-1} \)

1. \( n \) \( \pi_1 \cdots \pi_{n-1} \) \( \Rightarrow \) \( |\text{asc}(\pi')| = |\text{asc}(\pi)|, \quad \text{asc}(\pi') = \{ k \mid k - 1 \in \text{asc}(\pi) \} \)

2. \( \pi_1 \cdots \pi_{n-1} \) \( n \) \( \Rightarrow \) \( |\text{asc}(\pi')| = |\text{asc}(\pi)| + 1, \quad \text{asc}(\pi') = \text{asc}(\pi) \cup \{ n - 1 \} \)

3. \( \pi_1 \cdots \pi_i \) \( n \) \( \pi_{i+1} \cdots \pi_{n-1} \), where \( i \in \text{asc}(\pi) \), then \( |\text{asc}(\pi')| = |\text{asc}(\pi)| \), \n\( \text{asc}(\pi') = \{ k < i \mid k \in \text{asc}(\pi) \} \cup \{ k + 1 \mid k \in \text{asc}(\pi), \ k \geq i \} \)
Recursive construction of $\pi' = \pi'_1 \cdots \pi'_n$ from $\pi = \pi_1 \cdots \pi_{n-1}$

1. $\boxed{n} \pi_1 \cdots \pi_{n-1} \Rightarrow |\text{asc}(\pi')| = |\text{asc}(\pi)|$, $\text{asc}(\pi') = \{k \mid k - 1 \in \text{asc}(\pi)\}$

2. $\pi_1 \cdots \pi_{n-1} \boxed{n} \Rightarrow |\text{asc}(\pi')| = |\text{asc}(\pi)| + 1$, $\text{asc}(\pi') = \text{asc}(\pi) \cup \{n - 1\}$

3. $\pi_1 \cdots \pi_i \boxed{n} \pi_{i+1} \cdots \pi_{n-1}$, where $i \in \text{asc}(\pi)$,
   then $|\text{asc}(\pi')| = |\text{asc}(\pi)|$, $\text{asc}(\pi') = \{k < i \mid k \in \text{asc}(\pi)\} \cup \{k + 1 \mid k \in \text{asc}(\pi), k \geq i\}$

4. $\pi_1 \cdots \pi_i \boxed{n} \pi_{i+1} \cdots \pi_{n-1}$, where $i \in \text{des}(\pi)$,
   then $|\text{asc}(\pi')| = |\text{asc}(\pi)| + 1$, $\text{asc}(\pi') = \{k < i \mid k \in \text{asc}(\pi)\} \cup \{i\} \cup \{k + 1 \mid k \in \text{asc}(\pi), k \geq i\}$
Recursive construction of $\pi' = \pi'_1 \cdots \pi'_n$ from $\pi = \pi_1 \cdots \pi_{n-1}$

1. $n \pi_1 \cdots \pi_{n-1} \Rightarrow |asc(\pi')| = |asc(\pi)|, \quad asc(\pi') = \{k | k - 1 \in asc(\pi)\}$

2. $\pi_1 \cdots \pi_{n-1} n \Rightarrow |asc(\pi')| = |asc(\pi)| + 1, \quad asc(\pi') = asc(\pi) \cup \{n - 1\}$

3. $\pi_1 \cdots \pi_i n \pi_{i+1} \cdots \pi_{n-1}$, where $i \in asc(\pi)$, then $|asc(\pi')| = |asc(\pi)|$, $asc(\pi') = \{k < i | k \in asc(\pi)\} \cup \{k + 1 | k \in asc(\pi), k \geq i\}$

4. $\pi_1 \cdots \pi_i n \pi_{i+1} \cdots \pi_{n-1}$, where $i \in des(\pi)$, then $|asc(\pi')| = |asc(\pi)| + 1$, $asc(\pi') = \{k < i | k \in asc(\pi)\} \cup \{i\} \cup \{k + 1 | k \in asc(\pi), k \geq i\}$

5. $\pi_1 \cdots \pi_{i-1} n^\uparrow \pi_i \cdots \pi_{n-1}$ then $|asc(\pi')| = |asc(\pi)| + 1$, $asc(\pi') = \{k < i | k \in asc(\pi)\} \cup \{i\} \cup \{k + 1 | k \in asc(\pi), k \geq i\}$

Orli Herscovici (Georgia Tech)
Generalized permutations: construction

Recursive construction of $\pi' = \pi'_1 \cdots \pi'_n$ from $\pi = \pi_1 \cdots \pi_{n-1}$

1. $\square^n \pi_1 \cdots \pi_{n-1} \Rightarrow |asc(\pi')| = |asc(\pi)|, \quad asc(\pi') = \{k \mid k - 1 \in asc(\pi)\}$

2. $\pi_1 \cdots \pi_{n-1} \square^n \Rightarrow |asc(\pi')| = |asc(\pi)| + 1, \quad asc(\pi') = asc(\pi) \cup \{n - 1\}$

3. $\pi_1 \cdots \pi_i \square^n \pi_{i+1} \cdots \pi_{n-1}$, where $i \in asc(\pi)$,
   then $|asc(\pi')| = |asc(\pi)|$,
   $asc(\pi') = \{k < i \mid k \in asc(\pi)\} \cup \{k + 1 \mid k \in asc(\pi), k \geq i\}$

4. $\pi_1 \cdots \pi_i \square^n \pi_{i+1} \cdots \pi_{n-1}$, where $i \in des(\pi)$,
   then $|asc(\pi')| = |asc(\pi)| + 1$,
   $asc(\pi') = \{k < i \mid k \in asc(\pi)\} \cup \{i\} \cup \{k + 1 \mid k \in asc(\pi), k \geq i\}$

5. $\pi_1 \cdots \pi_{i-1} \uparrow^n \pi_i \cdots \pi_{n-1}$
   then $|asc(\pi')| = |asc(\pi)| + 1$,
   $asc(\pi') = \{k < i \mid k \in asc(\pi)\} \cup \{i\} \cup \{k + 1 \mid k \in asc(\pi), k \geq i\}$

6. $\downarrow^n$ does not change a number of existing ascents of permutations.
Generalized permutations with ascents: construction

Example

\[ n=1 \]

\[ 1 \]

\[ n=3 \] Possible tiles for insertion: \( 3, 3\uparrow, 3\downarrow \)
Example

n=1

n=2 Possible tiles for insertion: \(2\), \(\uparrow\), \(\downarrow\)
Example

n=1  \[1\]

n=2  Possible tiles for insertion:  \[2, 2↑, 2↓\]

Permutations:  \[1 2, 2 1, 2↑ 1, 2↓ 1\]
Example

n=1

n=2 Possible tiles for insertion: \(2, 2\uparrow, 2\downarrow\)

Permutations: 1 2, 2 1, 2\uparrow 1, 2\downarrow 1
Generalized permutations with ascents: construction

<table>
<thead>
<tr>
<th>Example</th>
<th>n=1</th>
<th>n=2 Possible tiles for insertion:</th>
<th>Permutations:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2, 2↑, 2↓</td>
<td>1, 2, 21, 2↑1, 2↓1</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>n=2</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>n=3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Generalized permutations with ascents: construction

Example

n=1

1

n=2

Possible tiles for insertion: 2, 2↑, 2↓

Permutations: 1 2, 2 1, 2↑ 1, 2↓ 1

n=3

Possible tiles for insertion: 3, 3↑, 3↓

Permutations: 1 2 3↑ 1 2, 1 3↑ 2, ...
Generalized permutations with ascents: construction

Example

$n=1$

\[
\begin{array}{c}
1
\end{array}
\]

$n=2$ Possible tiles for insertion: 

\[
\begin{array}{c}
2 \quad 2^\uparrow \quad 2^\downarrow
\end{array}
\]

Permutations:

\[
\begin{array}{c}
1 \quad 2 \quad , \quad 2 \quad 1 \quad , \quad 2^\uparrow \quad 1 \quad , \quad 2^\downarrow \quad 1
\end{array}
\]

$n=3$ Possible tiles for insertion: 

\[
\begin{array}{c}
3 \quad 3^\uparrow \quad 3^\downarrow
\end{array}
\]

Permutations:

\[
\begin{array}{c}
1 \quad 2 \quad 3 \quad ^\uparrow \quad 1 \quad 2 \quad , \quad 1 \quad 3 \quad ^\uparrow \quad 2 \quad , \quad 2 \quad ^\uparrow \quad 1 \quad 3 \quad ^\uparrow \quad 2 \quad , \quad , \quad \ldots
\end{array}
\]

\[
\begin{array}{c}
2^\uparrow \quad 1 \quad 3 \quad ^\uparrow \quad 2 \quad ^\uparrow \quad 1 \quad 2 \quad ^\uparrow \quad 3 \quad 1 \quad , \quad 2^\uparrow \quad 3 \quad ^\downarrow \quad 1 \quad , \quad \ldots
\end{array}
\]
Generalized permutations with ascents: construction

Example

n=1

\[
\begin{array}{c}
1
\end{array}
\]

n=2 Possible tiles for insertion: \[2, 2^{\uparrow}, 2^{\downarrow}\]

Permutations:

\[
\begin{array}{c}
1 & 2, & 2 & 1, & 2^{\uparrow} & 1,
\end{array}
\]

n=3 Possible tiles for insertion: \[3, 3^{\uparrow}, 3^{\downarrow}\]

Permutations:

\[
\begin{array}{c}
1 & 2, & 3^{\uparrow} & 1 & 2, & 1 & 3^{\uparrow} & 2, & \ldots
\end{array}
\]

\[
\begin{array}{c}
2^{\uparrow} & 1, & 3^{\uparrow} & 2^{\uparrow} & 1 & 2^{\uparrow} & 3 & 1, & 2^{\uparrow} & 3^{\downarrow} & 1, & \ldots
\end{array}
\]

\[
\begin{array}{c}
2^{\downarrow} & 1, & 2^{\downarrow} & 3 & 1, & \ldots
\end{array}
\]
Generalized permutations: construction

Example (Legal recursive construction)

Let us consider a recursive construction of a permutation $\pi = 2\downarrow, 4, 3\uparrow, 1$.

1: a trivial permutation $\pi = 1$ with no ascents.
2: $\pi = 2\downarrow, 1$ and w.r.t the Rule 6 the number of ascents is not changed.
3: $\pi = 2\downarrow, 3\uparrow, 1$. In accordance with the Rule 5 $asc(\pi) = \{2\}$ and $|asc(\pi)| = 1$.
4: $\pi = 2\downarrow, 4, 3\uparrow, 1$. In accordance with the Rule 3 $asc(\pi) = \{1, 3\}$ and $|asc(\pi)| = 2$.

Example (Illegal construction)

Let us consider another permutation $\pi = 2\downarrow, 1, 3\uparrow, 4$.

3: $\pi = 2\downarrow, 1, 3\uparrow$. Illegal w.r.t. the Rule 5.
Example (Legal recursive construction)

Let us consider a recursive construction of a permutation \( \pi = 2\downarrow, 4, 3\uparrow, 1 \).

1: a trivial permutation \( \pi = 1 \) with no ascents.

Example (Illegal construction)

Let us consider another permutation \( \pi = 2\downarrow, 1, 3\uparrow, 4 \).
Example (Legal recursive construction)

Let us consider a recursive construction of a permutation \( \pi = 2\downarrow, 4, 3\uparrow, 1 \).

1: a trivial permutation \( \pi = 1 \) with no ascents.

2\downarrow: \( \pi = 2\downarrow, 1 \) and w.r.t. the Rule 6 the number of ascents is not changed.
Example (Legal recursive construction)

Let us consider a recursive construction of a permutation
\( \pi = 2\downarrow, 4, 3\uparrow, 1. \)

1: a trivial permutation \( \pi = 1 \) with no ascents.

2\downarrow: \( \pi = 2\downarrow, 1 \) and w.r.t the Rule 6 the number of ascents is not changed.

3\uparrow: \( \pi = 2\downarrow, 3\uparrow, 1 \). In accordance with the Rule5 \( asc(\pi) = \{2\} \) and \( |asc(\pi)| = 1. \)
Example (Legal recursive construction)

Let us consider a recursive construction of a permutation \( \pi = 2\downarrow, 4, 3\uparrow, 1 \).

1: a trivial permutation \( \pi = 1 \) with no ascents.

2\downarrow: \( \pi = 2\downarrow, 1 \) and w.r.t the Rule 6 the number of ascents is not changed.

3\uparrow: \( \pi = 2\downarrow, 3\uparrow, 1 \). In accordance with the Rule5 \( \text{asc}(\pi) = \{2\} \) and \(|\text{asc}(\pi)| = 1\).

4: \( \pi = 2\downarrow, 4, 3\uparrow, 1 \). In accordance with the Rule3 \( \text{asc}(\pi) = \{1, 3\} \) and \(|\text{asc}(\pi)| = 2\).
Generalized permutations: construction

Example (Legal recursive construction)

Let us consider a recursive construction of a permutation $\pi = 2\downarrow, 4, 3\uparrow, 1$.

1: a trivial permutation $\pi = 1$ with no ascents.

2$\downarrow$: $\pi = 2\downarrow, 1$ and w.r.t the Rule 6 the number of ascents is not changed.

3$\uparrow$: $\pi = 2\downarrow, 3\uparrow, 1$. In accordance with the Rule5 $\text{asc}(\pi) = \{2\}$ and $|\text{asc}(\pi)| = 1$.

4: $\pi = 2\downarrow, 4, 3\uparrow, 1$. In accordance with the Rule3 $\text{asc}(\pi) = \{1, 3\}$ and $|\text{asc}(\pi)| = 2$.

Example (Illegal construction)

Let us consider another permutation $\pi = 2\downarrow, 1, 3\uparrow, 4$. ...
Generalized permutations: construction

Example (Legal recursive construction)
Let us consider a recursive construction of a permutation \( \pi = 2 \downarrow, 4, 3 \uparrow, 1 \).

1: a trivial permutation \( \pi = 1 \) with no ascents.

2\downarrow: \( \pi = 2 \downarrow, 1 \) and w.r.t the Rule 6 the number of ascents is not changed.

3\uparrow: \( \pi = 2 \downarrow, 3 \uparrow, 1 \). In accordance with the Rule5 \( \text{asc}(\pi) = \{2\} \) and \( |\text{asc}(\pi)| = 1 \).

4: \( \pi = 2 \downarrow, 4, 3 \uparrow, 1 \). In accordance with the Rule3 \( \text{asc}(\pi) = \{1, 3\} \) and \( |\text{asc}(\pi)| = 2 \).

Example (Illegal construction)
Let us consider another permutation \( \pi = 2 \downarrow, 1, 3 \uparrow, 4 \).

\[ \ldots \]

3\uparrow: \( \pi = 2 \downarrow, 1, 3 \uparrow \). **ILLEGAL** w.r.t. the Rule5
Generating function

Notations:
- \( GSA_n \) – a set of all generalized permutations of \([n]\) with ascents,
- \( \pi = \pi_1 \pi_2 \ldots \pi_n \in GSA_n \),
- \( \text{nua}(\pi) \) – a number of tiles with \( \uparrow \) in \( \pi \),
- \( \text{nda}(\pi) \) – a number of tiles with \( \downarrow \) in \( \pi \).
Generating function

Notations:
- $\text{GSA}_n$ – a set of all generalized permutations of $[n]$ with ascents,
- $\pi = \pi_1 \pi_2 \ldots \pi_n \in \text{GSA}_n$,
- $\text{nua}(\pi)$ – a number of tiles with $\uparrow$ in $\pi$,
- $\text{nda}(\pi)$ – a number of tiles with $\downarrow$ in $\pi$.

We denote by $A(n, k; u, d)$ a generating function for the generalized permutations of $[n]$ with exactly $k$ ascents according to the statistics $\text{nua}(\pi)$ and $\text{nda}(\pi)$, that is

$$A(n, k; u, d) = \sum_{\pi \in \text{GSA}_n} u^{\text{nua}(\pi)} d^{\text{nda}(\pi)}.$$
Notations:
- \( GSA_n \) – a set of all generalized permutations of \([n]\) with ascents,
- \( \pi = \pi_1 \pi_2 \ldots \pi_n \in GSA_n \),
- \( nua(\pi) \) – a number of tiles with \( \uparrow \) in \( \pi \),
- \( nda(\pi) \) – a number of tiles with \( \downarrow \) in \( \pi \).

We denote by \( A(n, k; u, d) \) a generating function for the generalized permutations of \([n]\) with exactly \( k \) ascents according to the statistics \( nua(\pi) \) and \( nda(\pi) \), that is

\[
A(n, k; u, d) = \sum_{\pi \in GSA_n} u^{nua(\pi)} d^{nda(\pi)}.
\]

We have

\[
A(n, k; u, d) = \begin{cases} 
\neq 0 & \text{for } 0 \leq k < n \text{ for all } n \in \mathbb{N}, \\
= 0 & \text{otherwise.}
\end{cases}
\]
Theorem (H., 2020) \((A336633)\)

The generating function \(A(n, k; u, d)\) satisfies the following recurrence relation

\[
A(n, k; u, d) = (k + 1 + (n - 1)d)A(n - 1, k; u, d) \\
+ (n - k + (n - 1)u)A(n - 1, k - 1; u, d),
\]

with initial condition \(A(1, 0; u, d) = 1\).

\[
A(n; u, d) = \sum_{k=0}^{n-1} A(n, k; u, d)x^k
\]

\(A(1; u, d) = 1\),

\(A(2; u, d) = (1 + d) + (1 + u)x\),

\(A(3; u, d) = (2d^2 + 3d + 1) + (4ud + 4u + 4d + 1)x + (2u^2 + 3u + 1)x^2\)
Theorem (H., 2020) (A336633)

The generating function $A(n, k; u, d)$ satisfies the following recurrence relation

$$A(n, k; u, d) = (k + 1 + (n - 1)d)A(n - 1, k; u, d)$$
$$+ (n - k + (n - 1)u)A(n - 1, k - 1; u, d),$$

with initial condition $A(1, 0; u, d) = 1$. 
Recurrence relation

**Theorem (H., 2020) (A336633)**

The generating function $A(n, k; u, d)$ satisfies the following recurrence relation

$$A(n, k; u, d) = (k + 1 + (n - 1)d)A(n - 1, k; u, d)$$
$$+ (n - k + (n - 1)u)A(n - 1, k - 1; u, d),$$

with initial condition $A(1, 0; u, d) = 1$.

**Reminder 1: degenerate Eulerian numbers of Carlitz (1979)**

$$A_{n,k}(\lambda) = (k + 1 - (n - 1)\lambda)A_{n-1,k}(\lambda) + (n - k + (n - 1)\lambda)A_{n-1,k-1}(\lambda).$$
Recurrence relation

**Theorem (H., 2020) (A336633)**

The generating function $A(n, k; u, d)$ satisfies the following recurrence relation

$$A(n, k; u, d) = (k + 1 + (n - 1)d)A(n - 1, k; u, d) + (n - k + (n - 1)u)A(n - 1, k - 1; u, d),$$

*with initial condition* $A(1, 0; u, d) = 1$.

**Reminder 1: degenerate Eulerian numbers of Carlitz (1979)**

$$A_{n,k}(\lambda) = (k + 1 - (n - 1)\lambda)A_{n-1,k}(\lambda) + (n - k + (n - 1)\lambda)A_{n-1,k-1}(\lambda).$$

**Corollary (Connection to Carlitz’s degenerate Eulerian numbers)**

$$A(n, k; \lambda, -\lambda) = A_{n,k}(\lambda).$$
Symmetry relation (Carlitz, 1979)

\[ A_{n,n-k+1}(\lambda) = A_{n,k}(-\lambda), \quad 1 \leq k \leq n \]
Symmetry relation

Symmetry relation (Carlitz, 1979)

\[ A_{n,n-k+1}(\lambda) = A_{n,k}(-\lambda), \quad 1 \leq k \leq n \]

Symmetry relation (H., 2020)

\[ A(n, k; u, d) = A(n, n - k - 1; d, u), \quad 0 \leq k \leq n - 1 \]
Reminder 2: degenerate Eulerian numbers of Carlitz (1979)

\[
\sum_{k=1}^{n} A_{n,k}(\lambda) = n!
\]

where \( A_{n,k}(\lambda) \) are the degenerate Eulerian numbers.
Reminder 2: degenerate Eulerian numbers of Carlitz (1979)

\[ \sum_{k=1}^{n} A_{n,k}(\lambda) = n! \]

Easy combinatorial proof by applying the proposed generalization of permutations!
Reminder 2: degenerate Eulerian numbers of Carlitz (1979)

\[
\sum_{k=1}^{n} A_{n,k}(\lambda) = n!
\]

Easy combinatorial proof by applying the proposed generalization of permutations!

Theorem (H., 2020, A016777)

\[
|GSA_n| = \prod_{k=1}^{n} (3k - 2) = \sum_{j=0}^{n} |s(n, n-j)| \cdot 3^j,
\]

where \(s(n, k)\) are the Stirling numbers of the first kind.
Thank you for your attention