Coloring Subsets by r-wise Intersecting Classes

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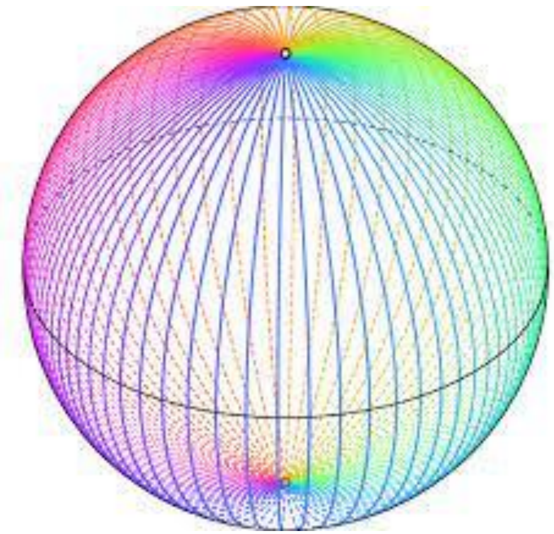

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I. The main conjecture

Question: **What is the minimum number of colors required to color all k-subsets of an n-set so that every color class is r-wise intersecting ?**

Definition: **A family of subsets is r-wise intersecting if every collection of at most r members of it has a common point.**

Note: one color suffices iff $k>(r-1)n/r$

Construction:

s = largest integer smaller than $\frac{rk}{r-1}$

For $i<n-s+1$, F_i = of all k-subsets A with min A=i

Color classes: these F_i and one more color class containing all remaining k-subsets

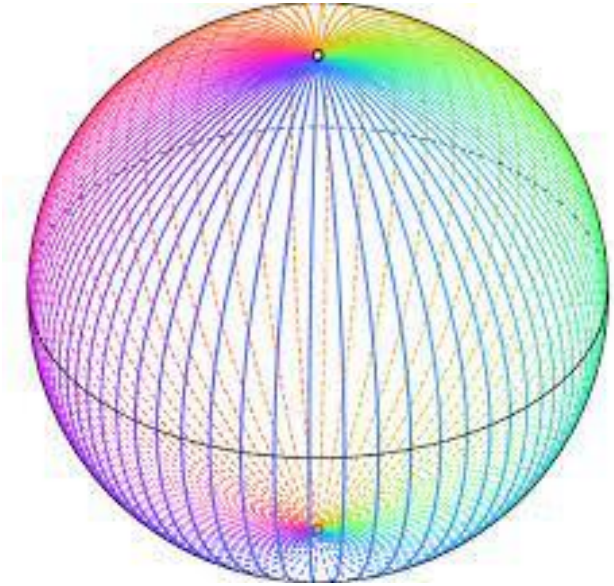
This gives that $\left\lceil n - \frac{r}{r-1}(k-1) \right\rceil$ colors suffice.

Conjecture: this is tight for all k ≤ (r-1)k/r.

Conjecture (equivalent formulation): If $n \geq (t-1)+kr/(r-1)$ then in any **t-coloring** of all k-subsets of an n-set $[n]=\{1,2,\ldots,n\}$ there are (at most) r k-subsets of the same color that do not share a common point.

Note: for r=2 this is **Kneser's conjecture (1955)** proved by **Lovász (1978)**

Simplified proofs: Bárány (1978), Greene (2002)
All proofs are topological, applying the Borsuk-Ulam Theorem.



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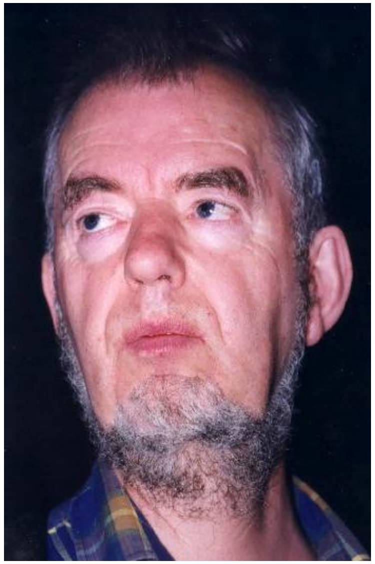
II. Tackling the conjecture: attempt no. 0

Apply tools from equivariant topology in the spirit of the Bárány-Shlosman-Szűcs (81) topological Tverberg's Theorem.



Approach appears to give an approximate version of the conjecture for some parameters

III. Tackling the conjecture: attempt no. 1

Try to reduce it to known results about Kneser Hypergraphs

Definition: The Kneser Hypergraph $KG^r(k,n)$ is the r-uniform hypergraph whose vertices are all k-subsets of [n], where an r-tuple of k- subsets forms an edge iff the subsets are pairwise disjoint.

Theorem [Alon, Frankl, Lovász (86), conjectured by Erdős (73)]: If $n \geq (t-1)(r-1)+kr$ then the chromatic number of $KG^r(k,n)$ is > t

Thm (AFL, equivalent formulation): if $n \geq (t-1)(r-1)+kr$ then in any **t-coloring** of all k-subsets of [n] there are r pairwise disjoint subsets of the same color.

Conjecture: if $n \geq (t-1)+kr/(r-1)$ then in any **t-coloring** of all k-subsets of [n] there are r subsets of the same color that do not share a common point.

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Attempted proof: Given such a t-coloring, replace each i in [n] by a set C_i of size r-1. For each k-subset $F=\{i_1, i_2, \ldots, i_k\}$, let C(F) be all $(r-1)^k$ subsets containing one element of each C_{i_j} and color all members of C(F) by the color of F.

This gives a t-coloring of (almost all) k-subsets of a set of size at least (t-1)(k-1)+kr. The hope is to use AFL and get r pairwise disjoint subsets of the same color here, which would give r subsets of the same color with no common point.

The trouble is that this is not a coloring of all k-subsets, the subsets containing more than one element in some C_i are absent. Thus we cannot use AFL, as we don't have here a coloring of the hypergraph $KG^r(k,n)$, only a coloring of an induced subgraph of it.

Wishful thinking: maybe this induced subgraph has the same chromatic number as the full hypergraph ?

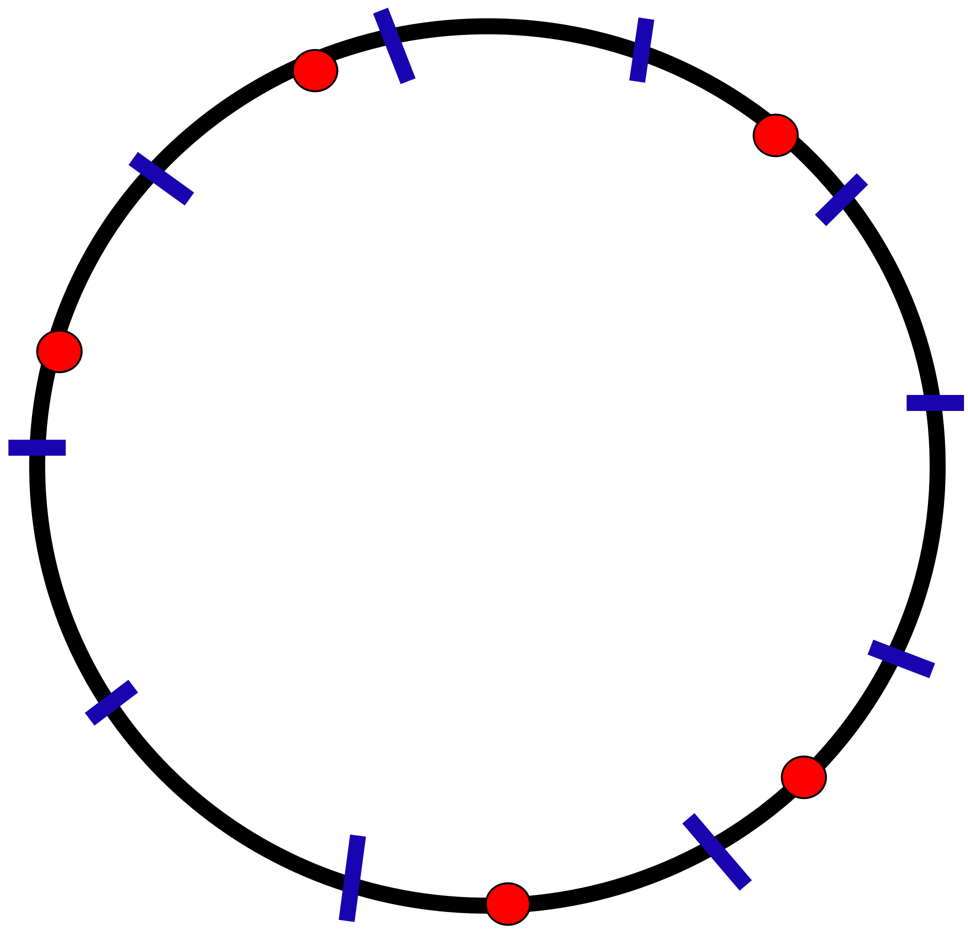
IV. Tackling the conjecture: attempt no. 2

Try to reduce it to known results about
Stable Kneser Hypergraphs

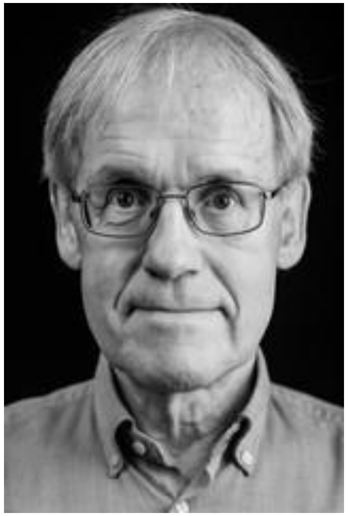
Definition: **A k-subset F of [n] is r-stable if any two elements of F are at distance at least r in the cyclic order on [n].**

The Stable Kneser Hypergraph $SKG^r(k,n)$ is the induced subgraph of $KG^r(k,n)$ whose vertices are all r-stable k-subsets of [n].

In the construction in attempt no. 1, if we place every set C_i contiguously along the cycle of length (r-1)n, all edges of $SKG^r(k,n)$ do appear



Conjecture [Ziegler (2002), Alon, Drewnowski, Łuczak (2009)]: The chromatic number of $SKG^r(k,n)$ is equal to that of the full hypergraph $KG^r(k,n)$.



This holds for r=2, as proved by Schrijver (1978)

In ADL it is shown that if it holds for r_1 and r_2 , it also holds for $r=r_1r_2$. Thus it holds for any r which is a power of 2.

This is used in ADL to construct ideals of natural numbers which are not nonatomic yet have the Nykodým property.

It gives unexpected examples of ideals with the positive summability property, settling a problem of Drewnowski and Paul (2000).

The relevant property of stable Kneser hypergraphs is the following: For every integer $r \geq 2$, any (small) $\epsilon > 0$ and any (large) C, there exists a stable Kneser r-uniform hypergraph K=(V,E) with chromatic number > C satisfying the following: For any weight function w(v) on the vertices and for any $1 \leq s < r$, there exists a subset W of V containing at most s vertices of any edge, so that

$$\sum_{v \in W} w(v) \geq (1-\epsilon)\,\frac{s}{r} \sum_{v \in V} w(v)$$

The Ziegler+ADL conjecture is open for all r which is not a power of 2, but a weaker result of Frick (2020) about this conjecture suffices to prove the conjecture discussed here for every prime r.

Conjecture [Ziegler, ADL]: The chromatic number of $SKG^r(k,n)$ is equal to that of the full hypergraph $KG^r(k,n)$. Equivalently: if $n \geq (t-1)(r-1)+kr$, then in any t-coloring of the r-stable k-subsets of [n] there are r pairwise disjoint r-stable subsets of the same color.

The argument that if it holds for r_1 and r_2 , it also holds for $r=r_1r_2$:

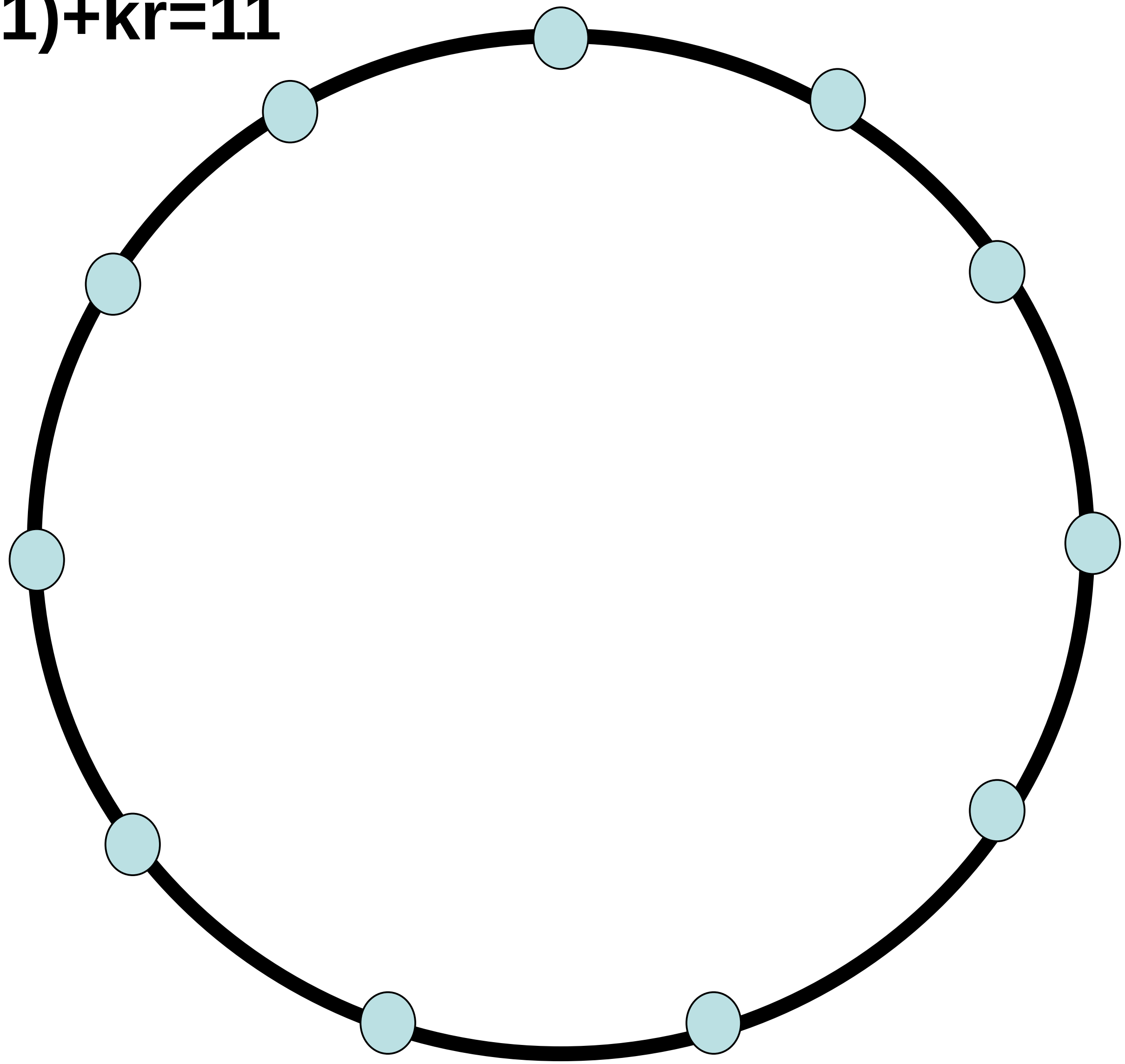
Given a t-coloring of the $r=r_1r_2$ –stable k-subsets of [n], put $k_1=(t-1)(r_1-1)+kr_1$ and define a t-coloring of the r_2-stable k_1-subsets F of [n] as follows.

Every r_1-stable k-subset of F is $r=r_1r_2$ stable in [n], hence has a color. By the result for r_1 there are r_1 pairwise disjoint r_1-stable k-subsets of F of some color j, color F by this color j.

Note that $n \geq (t-1)(r_1r_2-1)+kr_1r_2=(t-1)(r_2-1)+k_1r_2$ = $(t-1)(r_2-1)+[(t-1)(r_1-1)+kr_1]r_2$ (!)

By the result for r_2 there are r_2 pairwise disjoint r_2 –stable k_1-subsets of the same color j. Each of them contains r_1 pairwise disjoint r_1-stable k-subsets of the same color j, giving altogether $r=r_1r_2$ pairwise disjoint r_1r_2-stable k-subsets of color j in the original coloring.

Example: $t=r_1=r_2=k=2$, $r=r_1r_2=4$, $k_1=(t-1)(r_1-1)+kr_1=5$
$n=(t-1)(r-1)+kr=11$



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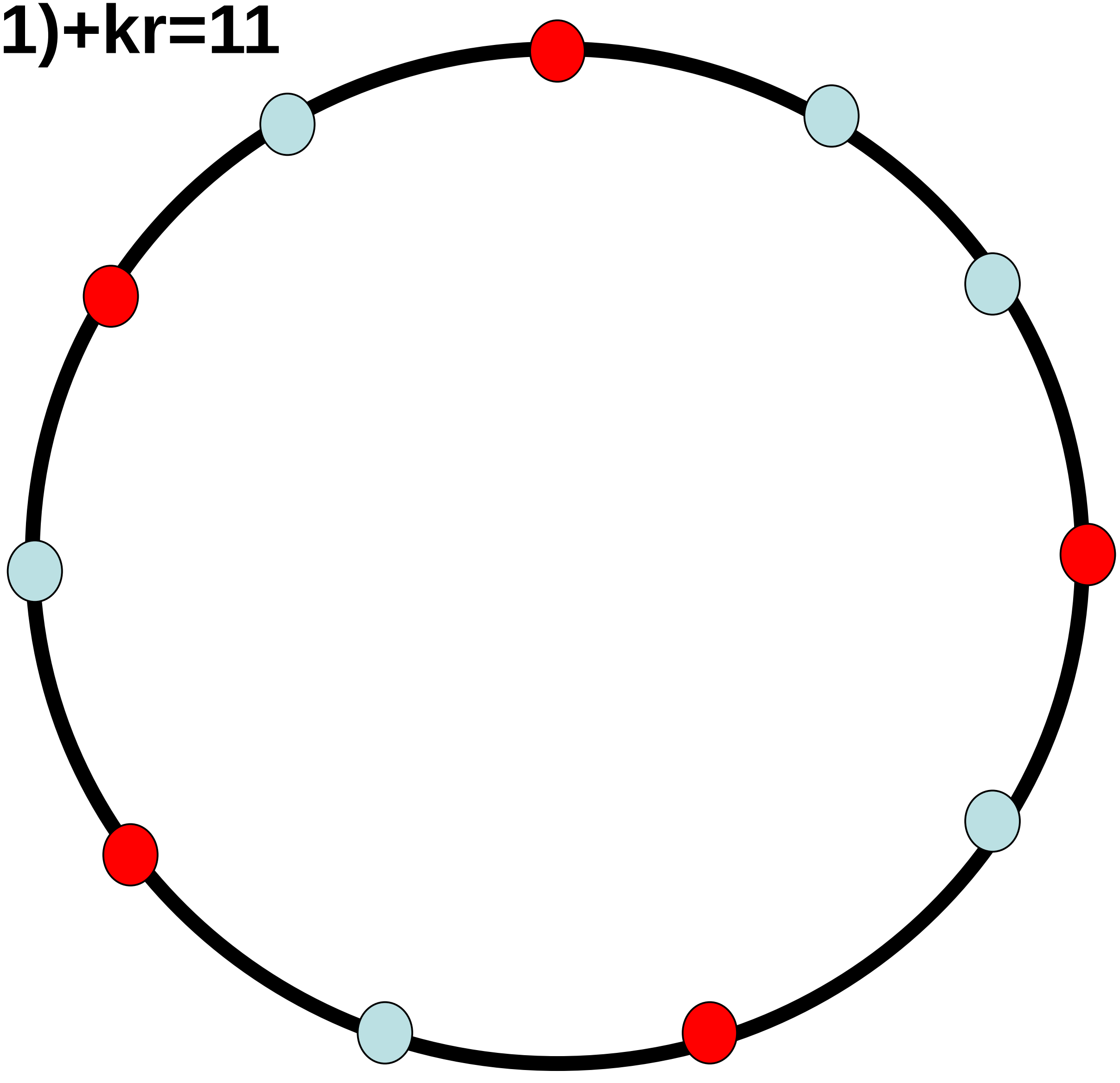

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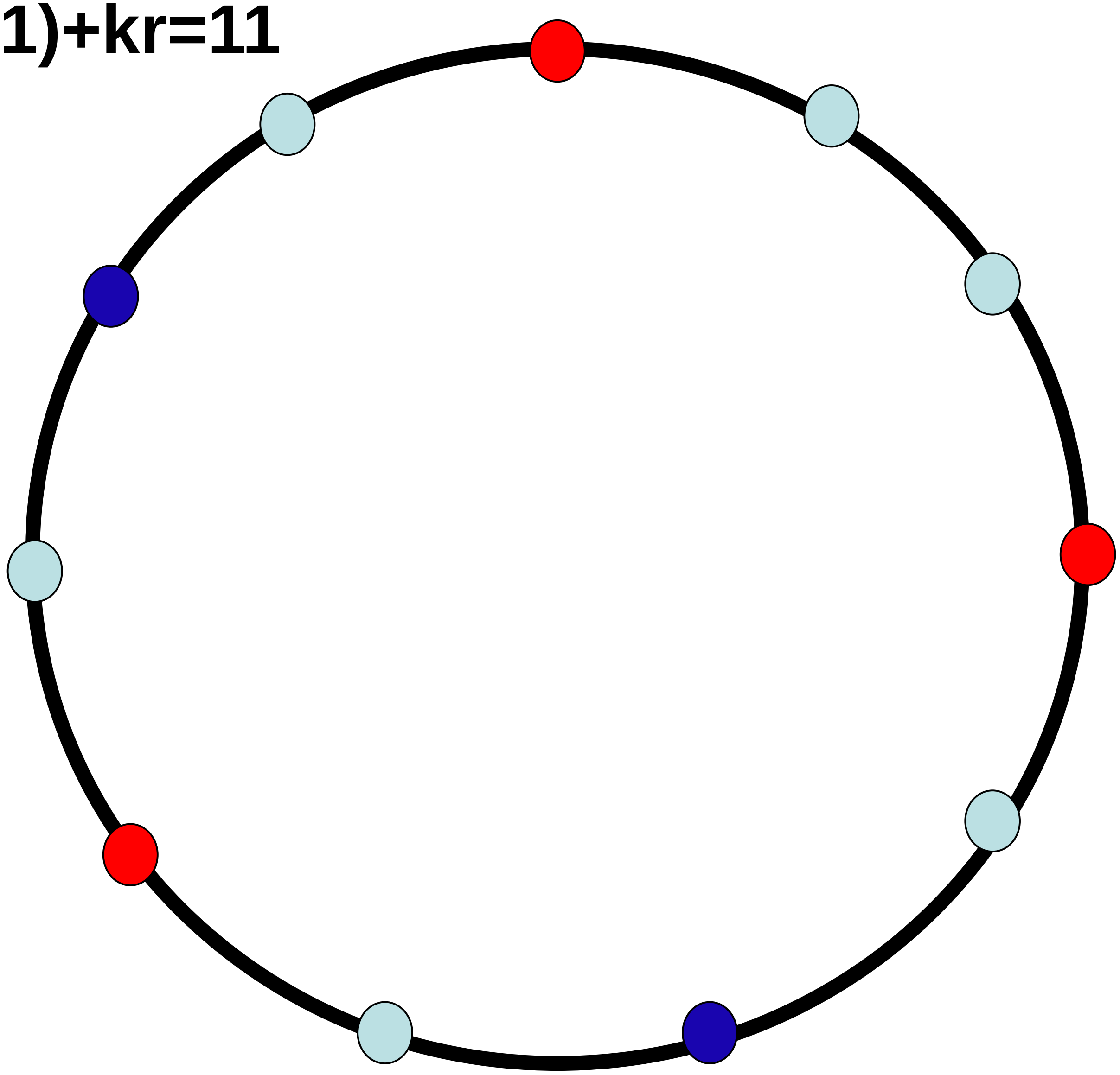

V. Remarks, Problems

Does the conjecture hold for all admissible values of the parameters t, k, r ?

Remark: easy to see it holds for r>k (as in this case every r-wise intersecting family which does not contain a common point [is not a star] is of size at most k, hence if we have less than t-1 stars we can't cover all k-subsets)

Problem (with **Ryan Alweiss**): Let V be a linear **space** of dimension m of vectors of length n over F_2 and let F be the family of 2^m -1 subsets of [n] whose characteristic vectors are the nonzero members of V. Can F be partitioned into o(m) **3-wise intersecting** families ?

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Remarks: By the validity of the main conjecture for r=3 the answer is NO if n is at most 2.999 m

If the answer is NO for every such V with n=20 m, then in any coloring of the elements of the group F_2^m by o(m) colors there is a monochromatic solution to the equation x+y=z (This is Schur's Problem for F_2^m)

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If the answer is YES for some such V and some n, then the t-colors Ramsey number r(3,3,…,3) is larger than exponential in t and hence the maximum possible Shannon Capacity of a graph with independence number 2 is not finite (this is the Schur-Erdős Problem).

Note that 0.75 m colors suffice using some properties of the Clebsch graph

Thank You