A Computational Approach to Improving Bounds on the Hales–Jewett Numbers

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The classic two-dimensional game of tic-tac-toe is strategically trivial, as optimal play typically results in a draw. In higher dimensions, however, the Hales–Jewett theorem guarantees that for any fixed number of players and board size, there exists a finite dimension beyond which a tie is impossible. The *Hales–Jewett number $HJ(k; r)$* is defined as the smallest such dimension for a k-in-a-row game with r players. Despite its conceptual simplicity, determining Hales–Jewett numbers is notoriously difficult due to the combinatorial explosion of possible game outcomes; in the more than fifty years since the theorem's proof, only one nontrivial Hales–Jewett number has been computed exactly.

In this work, we develop a new computational approach that combines SAT solvers with van der Waerden–based constructions to improve bounds on the undetermined numbers $HJ(3; 3)$ and $HJ(4; 2)$. Using this method, we establish an improved lower bound of $HJ(3; 3) \geq 14$, which is the first known bound to exceed that suggested by van der Waerden's theorem. In addition, we extend our approach to obtain new results for an asymmetric variant of the problem, known as the off-diagonal Hales–Jewett numbers.