The classical radiation-reaction problem

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Microscopic classical foundations of relativistic physics

WANTED:

A well-posed *generally covariant* joint initial value problem for:

The motion of massive charged point particles and the evolution of the electromagnetic and gravitational fields they generate.

The Classical Radiation-Reaction Problem has been in the way!

TODAY:

The Lorentz-covariant electromagnetic problem.
(Turning Gravity off: $G = 0$)

The failed proto-type: Lorentz electrodynamics

The Maxwell–Lorentz Field equations:

- The evolution equations for the fields:

$$\partial_t \mathbf{B}(t, \mathbf{s}) = -\nabla \times \mathbf{E}(t, \mathbf{s})$$
$$\partial_t \mathbf{E}(t, \mathbf{s}) = +\nabla \times \mathbf{B}(t, \mathbf{s}) - 4\pi \sum_k e_k \dot{\mathbf{q}}_k(t) \delta_{\mathbf{q}_k(t)}(\mathbf{s}),$$

- The constraint equations for the fields:

$$\nabla \cdot \mathbf{B}(t, \mathbf{s}) = 0$$
$$\nabla \cdot \mathbf{E}(t, \mathbf{s}) = 4\pi \sum_k e_k \delta_{\mathbf{q}_k(t)}(\mathbf{s})$$

- N.B.: Constraint equations restrict only field data.

The failed proto-type: Lorentz electrodynamics

The relativistic equations of particle motion

- Einstein–Lorentz–Poincaré velocity-momentum relation

$$\dot{\mathbf{q}}_k(t) = \frac{1}{m_k}\frac{\mathbf{p}_k(t)}{\sqrt{1+\frac{|\mathbf{p}_k(t)|^2}{m_k^2}}}; \quad m_k \neq 0$$

- Newton's law for the rate of change of momentum

$$\dot{\mathbf{p}}_k(t) = \mathbf{f}_k(t)$$

- Lorentz' law for the electromagnetic force

$$\mathbf{f}_k^{\mathrm{Lor}}(t) = e_k\left[\mathbf{E}(t,\mathbf{q}_k(t)) + \dot{\mathbf{q}}_k(t)\times\mathbf{B}(t,\mathbf{q}_k(t))\right]$$

The failed proto-type: Lorentz electrodynamics

Lorentz Electrodynamics is *not well-definable*!

- Symbolically the equations of Lorentz Electrodynamics seem to pose a joint Cauchy problem for positions $\mathbf{q}_k(t)$ and momenta $\mathbf{p}_k(t)$, and for the fields $\mathbf{B}(t,\mathbf{s})$ and $\mathbf{E}(t,\mathbf{s})$, with initial data constrained by the divergence equations.
- However, this Cauchy problem is ill defined!
- Reason: $\mathbf{E}(t,\mathbf{q}_k(t))$ and $\mathbf{B}(t,\mathbf{q}_k(t))$ "infinite in all directions"
- $\mathbf{f}_k^{\text{Lor}}(t)$ can be "defined" through averaging (very popular!), but result depends on how the averaging is done.
- Deckert and Hartenstein: Singularities on initial light cones.
- Also, fields too strongly divergent at particle world lines $\longrightarrow$

No meaningful energy-momentum conservation law!

Poincaré

The pre-metric Maxwell–Lorentz field equations

- Minkowski spacetime threaded by N timelike world-lines.
- Lorentz frame with space vector $\mathbf{s} \in \mathbb{R}^3$ and time $t \in \mathbb{R}$
- The evolution equations for the **B**, **D** fields

$$\partial_t \mathbf{B}(t,\mathbf{s}) = -\nabla \times \mathbf{E}(t,\mathbf{s})$$

$$\partial_t \mathbf{D}(t,\mathbf{s}) = +\nabla \times \mathbf{H}(t,\mathbf{s}) - 4\pi \sum_{k=1}^{N} e_k \dot{\mathbf{q}}_k(t) \delta_{\mathbf{q}_k(t)}(\mathbf{s})$$

- The constraint equations for the **B**, **D** fields

$$\nabla \cdot \mathbf{B}(t,\mathbf{s}) = 0$$

$$\nabla \cdot \mathbf{D}(t,\mathbf{s}) = 4\pi \sum_{k=1}^{N} e_k \delta_{\mathbf{q}_k(t)}(\mathbf{s})$$

- The constraint for the sources: subluminal velocities

$$|\dot{\mathbf{q}}_k(t)| < 1$$

Electromagnetic Vacuum Laws

Electromagnetic Vacuum Laws: $(\mathbf{B}, \mathbf{D}) \leftrightarrow (\mathbf{H}, \mathbf{E})$

- Maxwell(-Lorentz)'s law

$$\mathbf{H} = \mathbf{B} \qquad \& \qquad \mathbf{E} = \mathbf{D}$$

- Born-Infeld's law

$$\mathbf{H} = \frac{\mathbf{B} - \frac{1}{b^2}\mathbf{D} \times (\mathbf{D} \times \mathbf{B})}{\sqrt{1 + \frac{1}{b^2}(|\mathbf{B}|^2 + |\mathbf{D}|^2) + \frac{1}{b^4}|\mathbf{B} \times \mathbf{D}|^2}}$$

$$\mathbf{E} = \frac{\mathbf{D} - \frac{1}{b^2}\mathbf{B} \times (\mathbf{B} \times \mathbf{D})}{\sqrt{1 + \frac{1}{b^2}(|\mathbf{B}|^2 + |\mathbf{D}|^2) + \frac{1}{b^4}|\mathbf{B} \times \mathbf{D}|^2}}$$

- Bopp-Landé-Thomas(-Podolsky) law (N.B.: $\square := \partial_t^2 - \Delta$)

$$\mathbf{H}(t, \mathbf{s}) = \left(1 + \varkappa^{-2}\square\right)\mathbf{B}(t, \mathbf{s})$$

$$\mathbf{D}(t, \mathbf{s}) = \left(1 + \varkappa^{-2}\square\right)\mathbf{E}(t, \mathbf{s})\,.$$

ML and MBI and MBLTP Electromagnetic Field Theory

Rigorous Results on the Field Cauchy Problems

- ML field Cauchy problem (standard):
  Global well-posedness (weak) with "arbitrary" data.
- MBLTP field Cauchy problem (standard):
  Global well-posedness (weak) with "arbitrary" data.
- MBI field Cauchy problem:
  - Global well-posedness (classical) with small data (no charges!) (J. Speck; F. Pasqualotto)
  - Finite-time blow up with certain plane wave data (no charges!) (Y. Brenier; cf. D. Serre)
  - Existence and Uniqueness of static finite-energy solutions with N fixed point charges; real analyticity away from point charges (M.K.; cf. Bonheure et al.)

ML and MBI and MBLTP Electromagnetic Field Theory

MBLTP (and MBI ?) field momenta are finite!

- Field momentum density: $\mathbf{\Pi}$
- For ML and for MBI field equations

$$4\pi\mathbf{\Pi} = \mathbf{D} \times \mathbf{B}$$

- For MBLTP field equations

$$4\pi\mathbf{\Pi} = \mathbf{D} \times \mathbf{B} + \mathbf{E} \times \mathbf{H} - \mathbf{E} \times \mathbf{B} - \varkappa^{-2}\big(\nabla \cdot \mathbf{E}\big)\big(\nabla \times \mathbf{B} - \varkappa\dot{\mathbf{E}}\big)$$

- $\mathbf{\Pi}(t, \mathbf{s})$ is $L^1_{loc}(\mathbb{R}^3)$ about each $\mathbf{q}(t)$ for MBLTP fields (KTZ). (Expected for MBI fields, but surely FALSE for ML fields!)

BLTP Electrodynamics

Momentum Conservation $\longrightarrow$ Equation of Motion (here: 1 pt charge)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p}(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \mathbf{\Pi}(t,\mathbf{s})d^3s \qquad (*)$$

- With BLTP law: The fields $\mathbf{B}, \mathbf{D}, \mathbf{E}, \dot{\mathbf{E}}$ (and $\mathbf{H}$) at $(t,\mathbf{s})$ depend on $\mathbf{q}(\cdot)$, $\mathbf{p}(\cdot)$, and $\mathbf{D}$ & $\mathbf{H}$ also on $\mathbf{a}(\cdot)$.
- (*) is equivalent to Volterra integral equation for $\mathbf{a} = \mathbf{a}[\mathbf{q},\mathbf{p}]$
- This leads to the fixed point equations

$$\mathbf{q}(t) = \mathbf{q}(0) + \frac{1}{m}\int_0^t \frac{\mathbf{p}}{\sqrt{1+\frac{|\mathbf{p}|^2}{m^2}}}(\tilde{t})d\tilde{t} \qquad =: Q_t(\mathbf{q}(\cdot),\mathbf{p}(\cdot))$$

$$\mathbf{p}(t) = \mathbf{p}(0) - \int_{\mathbb{R}^3}\left(\mathbf{\Pi}(t,\mathbf{s}) - \mathbf{\Pi}(0,\mathbf{s})\right)d^3s =: P_t(\mathbf{q}(\cdot),\mathbf{p}(\cdot))$$

- Well-posedness if $(Q_\bullet, P_\bullet)(\,\cdot\,,\,\cdot\,)$ is a Lipschitz Map.

Thm: BLTP Electrodynamics is well-posed! (KTZ)

- The Cauchy problem for MBLTP field + N point charges:
  - Local well-posedness for admissible initial data & $m \neq 0$.
  - Global well-posedness if in a finite time:
    (a) no particle reaches the speed of light,
    (b) no particle reaches infinite acceleration,
    (c) no two particles reach the same location.
- Energy-Momentum conservation rigorously true.
- "Self"-force analyzed rigorously (cf. Hoang & Radosz)
- MBLTP oddities:
  (a) longitudinal electrical waves;
  (b) subluminal transversal electromagnetic wave modes;
  (c) energy functional unbounded below;
  (d) The MBLTP fields $\mathbf{B}, \mathbf{D}, \mathbf{E}, \dot{\mathbf{E}}$ require initial data.
  N.B.: $(\mathbf{B}, \mathbf{D})_0 \mapsto (\mathbf{E}, \dot{\mathbf{E}})_0$ feasible! (max. field energy)

The Volterra equation for the BLTP acceleration

$$\mathbf{a} = W[\mathbf{p}] \cdot \left(\mathbf{f}^{\text{vac}}[\mathbf{q}, \mathbf{v}] + \mathbf{f}^{\text{source}}[\mathbf{q}, \mathbf{v}; \mathbf{a}] \right)$$

where

$$\mathbf{v} = \frac{1}{m} \frac{\mathbf{p}}{\sqrt{1 + \frac{|\mathbf{p}|^2}{m^2}}}; \qquad m \neq 0$$

and

$$W[\mathbf{p}] := \operatorname{sign}(m) \frac{1}{\sqrt{m^2 + |\mathbf{p}|^2}} \left[\mathbf{I} - \frac{\mathbf{p} \otimes \mathbf{p}}{m^2 + |\mathbf{p}|^2} \right]$$

and

$$\mathbf{f}^{\text{vac}}[\mathbf{q}, \mathbf{v}](t) \equiv e \left[\mathbf{E}^{\text{vac}}(t, \mathbf{q}(t)) + \mathbf{v}(t) \times \mathbf{B}^{\text{vac}}(t, \mathbf{q}(t)) \right]$$

and .... DRUM ROLL.......

The Volterra equation for the acceleration

$\mathbf{f}^{\text{source}}[\mathbf{q}, \mathbf{v}; \mathbf{a}]$ is the "self" force in BLTP electrodynamics

$$\mathbf{f}^{\text{source}}[\mathbf{q}, \mathbf{v}; \mathbf{a}](t) = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \mathbf{\Pi}^{\text{source}}(t, \mathbf{s})d^3s$$

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$$= -\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_{ct}(\mathbf{q}_0)} \left(\mathbf{\Pi}(t,\mathbf{s}) - \mathbf{\Pi}(0,\mathbf{s}-\mathbf{q}_0-\mathbf{v}_0 t)\right) d^3s$$

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$$= -\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_{ct}(\mathbf{q}_0)}\left(\mathbf{\Pi}(t,\mathbf{s}) - \mathbf{\Pi}(0,\mathbf{s}-\mathbf{q}_0-\mathbf{v}_0 t)\right)d^3s$$

$$= \frac{e^2}{4\pi}\Bigg[-\mathbf{Z}^{[2]}_{\boldsymbol{\xi}}(t,t) + \mathbf{Z}^{[2]}_{\boldsymbol{\xi}^\circ}(t,t)$$

$$-\sum_{0\leq k\leq 1}(2-k)\int_0^t\left[\mathbf{Z}^{[k]}_{\boldsymbol{\xi}}(t,t^{\mathrm{r}}) - \mathbf{Z}^{[k]}_{\boldsymbol{\xi}^\circ}(t,t^{\mathrm{r}})\right](t-t^{\mathrm{r}})^{1-k}\mathrm{d}t^{\mathrm{r}}$$

$$-\sum_{0\leq k\leq 2}\int_0^t\left[\frac{\partial}{\partial t}\mathbf{Z}^{[k]}_{\boldsymbol{\xi}}(t,t^{\mathrm{r}}) - \frac{\partial}{\partial t}\mathbf{Z}^{[k]}_{\boldsymbol{\xi}^\circ}(t,t^{\mathrm{r}})\right](t-t^{\mathrm{r}})^{2-k}\mathrm{d}t^{\mathrm{r}}\Bigg]$$

where $\xi(t)\equiv(\mathbf{q},\mathbf{v},\mathbf{a})(t)$, and $\xi^\circ(t)\equiv(\mathbf{q}_0+\mathbf{v}_0 t,\mathbf{v}_0,\mathbf{0})$, and ...

The Volterra equation for the acceleration

$$\mathbf{Z}_{\boldsymbol{\xi}}^{[k]}(t, t^{\mathrm{r}}) =$$
$$\int_0^{2\pi}\int_0^{\pi} (1 - |\boldsymbol{v}(t^{\mathrm{r}})| \cos\vartheta)\, \boldsymbol{\pi}_{\boldsymbol{\xi}}^{[k]}\big(t, \mathbf{q}(t^{\mathrm{r}}) + c(t - t^{\mathrm{r}})\boldsymbol{n}\big) \sin\vartheta \mathrm{d}\vartheta \mathrm{d}\varphi\,,$$

with

$$\boldsymbol{n} = \begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix}$$

and where, with $\big|_{\mathrm{ret}}$ meaning that $\mathbf{q}(\tilde{t})$, $\boldsymbol{v}(\tilde{t})$, $\boldsymbol{a}(\tilde{t})$ are evaluated at $\tilde{t} = t_{\boldsymbol{\xi}}^{\mathrm{ret}}(t, \mathbf{s})$, we have ...

The Volterra equation for the acceleration

$$
\begin{aligned}
\pi^{[0]}_{\xi}(t,\mathbf{s}) = & - \varkappa^4 \frac{1}{4}\left[\frac{\big(\boldsymbol{n}(\mathbf{q},\mathbf{s}) - \boldsymbol{v}\big)\times\big(\boldsymbol{v}\times\boldsymbol{n}(\mathbf{q},\mathbf{s})\,\big)}{\big(1-\boldsymbol{v}\cdot\boldsymbol{n}(\mathbf{q},\mathbf{s})\big)^2}\right]_{\mathrm{ret}} \\
& + \varkappa^4 \frac{1}{2}\left[\frac{\boldsymbol{n}(\mathbf{q},\mathbf{s}) - \boldsymbol{v}}{1-\boldsymbol{v}\cdot\boldsymbol{n}(\mathbf{q},\mathbf{s})}\right]_{\mathrm{ret}} \times \int_{-\infty}^{t^{\mathrm{ret}}_{\xi}(t,\mathbf{s})} \boldsymbol{v}(t')\times\mathbf{K}_{\xi}(t',t,\mathbf{s})\mathrm{d}t' \\
& - \varkappa^4 \frac{1}{2}\left[\frac{\boldsymbol{v}\times\boldsymbol{n}(\mathbf{q},\mathbf{s})}{1-\boldsymbol{v}\cdot\boldsymbol{n}(\mathbf{q},\mathbf{s})}\right]_{\mathrm{ret}} \times \int_{-\infty}^{t^{\mathrm{ret}}_{\xi}(t,\mathbf{s})} \mathbf{K}_{\xi}(t',t,\mathbf{s})\mathrm{d}t' \\
& - \varkappa^4 \int_{-\infty}^{t^{\mathrm{ret}}_{\xi}(t,\mathbf{s})} \mathbf{K}_{\xi}(t',t,\mathbf{s})\mathrm{d}t' \times \int_{-\infty}^{t^{\mathrm{ret}}_{\xi}(t,\mathbf{s})} \boldsymbol{v}(t')\times\mathbf{K}_{\xi}(t',t,\mathbf{s})\mathrm{d}t' \\
& - \varkappa^4 \int_{-\infty}^{t^{\mathrm{ret}}_{\xi}(t,\mathbf{s})} \mathrm{K}_{\xi}(t',t,\mathbf{s})\mathrm{d}t' \int_{-\infty}^{t^{\mathrm{ret}}_{\xi}(t,\mathbf{s})} \mathrm{K}_{\xi}(t',t,\mathbf{s})\boldsymbol{v}(t')\mathrm{d}t'
\end{aligned}
$$

The Volterra equation for the acceleration

$$\pi_{\xi}^{[1]}(t,\mathbf{s}) = -\varkappa^2\left[\boldsymbol{n}(\mathbf{q},\mathbf{s})\frac{(\boldsymbol{n}(\mathbf{q},\mathbf{s})\times[(\boldsymbol{n}(\mathbf{q},\mathbf{s})-\boldsymbol{v})\times\boldsymbol{a}])\cdot\boldsymbol{v}}{\big(1-\boldsymbol{v}\cdot\boldsymbol{n}(\mathbf{q},\mathbf{s})\big)^4}+\boldsymbol{n}(\mathbf{q},\mathbf{s})\times\frac{\big(\boldsymbol{n}(\mathbf{q},\mathbf{s})-\boldsymbol{v}\big)\times\boldsymbol{a}}{2\big(1-\boldsymbol{v}\cdot\boldsymbol{n}(\mathbf{q},\mathbf{s})\big)^3}\right]_{\mathrm{ret}}$$

$$-\varkappa^2\left[\boldsymbol{n}(\mathbf{q},\mathbf{s})\times\frac{\big(\boldsymbol{n}(\mathbf{q},\mathbf{s})-\boldsymbol{v}\big)\times\boldsymbol{a}}{\big(1-\boldsymbol{v}\cdot\boldsymbol{n}(\mathbf{q},\mathbf{s})\big)^3}\right]_{\mathrm{ret}}\times\int_{-\infty}^{t_{\xi}^{\mathrm{ret}}(t,\mathbf{s})}\boldsymbol{v}(t')\times\mathbf{K}_{\xi}(t',t,\mathbf{s})\mathrm{d}t'$$

$$+\varkappa^2\left[\boldsymbol{n}(\mathbf{q},\mathbf{s})\times\left[\boldsymbol{n}(\mathbf{q},\mathbf{s})\times\frac{\big(\boldsymbol{n}(\mathbf{q},\mathbf{s})-\boldsymbol{v}\big)\times\boldsymbol{a}}{\big(1-\boldsymbol{v}\cdot\boldsymbol{n}(\mathbf{q},\mathbf{s})\big)^3}\right]\right]_{\mathrm{ret}}\times\int_{-\infty}^{t_{\xi}^{\mathrm{ret}}(t,\mathbf{s})}\mathbf{K}_{\xi}(t',t,\mathbf{s})\mathrm{d}t'$$

$$+\varkappa^3\left[\frac{1}{1-\boldsymbol{v}\cdot\boldsymbol{n}(\mathbf{q},\mathbf{s})}\right]_{\mathrm{ret}}\int_{-\infty}^{t_{\xi}^{\mathrm{ret}}(t,\mathbf{s})}\mathrm{K}_{\xi}(t',t,\mathbf{s})\left[\boldsymbol{v}(t_{\xi}^{\mathrm{ret}}(t,\mathbf{s}))+\boldsymbol{v}(t')\right]\mathrm{d}t'$$

The Volterra equation for the acceleration

$$
\begin{aligned}
\pi_\xi^{[2]}(t,s) = &- \varkappa^2 \left[\frac{1}{\big(1 - \boldsymbol{v}\cdot\boldsymbol{n}(\mathbf{q},\mathbf{s})\big)^2}\boldsymbol{v} - \Big[1 - |\boldsymbol{v}|^2\Big] \frac{\big(\boldsymbol{n}(\mathbf{q},\mathbf{s}) - \boldsymbol{v}\big)\times\big(\boldsymbol{v}\times\boldsymbol{n}(\mathbf{q},\mathbf{s})\big)}{\big(1 - \boldsymbol{v}\cdot\boldsymbol{n}(\mathbf{q},\mathbf{s})\big)^4} \right]_{\mathrm{ret}} \\
&+ \varkappa^2 \left[\Big[1 - |\boldsymbol{v}|^2\Big] \boldsymbol{n}(\mathbf{q},\mathbf{s}) \times \frac{\boldsymbol{n}(\mathbf{q},\mathbf{s}) - \boldsymbol{v}}{\big(1 - \boldsymbol{v}\cdot\boldsymbol{n}(\mathbf{q},\mathbf{s})\big)^3} \right]_{\mathrm{ret}} \times \int_{-\infty}^{t_\xi^{\mathrm{ret}}(t,\mathbf{s})} \mathbf{K}_\xi(t',t,\mathbf{s})\,\mathrm{d}t' \\
&- \varkappa^2 \left[\Big[1 - |\boldsymbol{v}|^2\Big] \frac{\boldsymbol{n}(\mathbf{q},\mathbf{s}) - \boldsymbol{v}}{\big(1 - \boldsymbol{v}\cdot\boldsymbol{n}(\mathbf{q},\mathbf{s})\big)^3} \right]_{\mathrm{ret}} \times \int_{-\infty}^{t_\xi^{\mathrm{ret}}(t,\mathbf{s})} \boldsymbol{v}(t')\times\mathbf{K}_\xi(t',t,\mathbf{s})\,\mathrm{d}t',
\end{aligned}
$$

with the abbreviations

$$
\mathrm{K}_\xi(t',t,\mathbf{s}) := \frac{J_1\Big(\varkappa\sqrt{(t-t')^2 - |\mathbf{s}-\mathbf{q}(t')|^2}\Big)}{\sqrt{(t-t')^2 - |\mathbf{s}-\mathbf{q}(t')|^2}},
$$

$$
\mathbf{K}_\xi(t',t,\mathbf{s}) := \frac{J_2\Big(\varkappa\sqrt{(t-t')^2 - |\mathbf{s}-\mathbf{q}(t')|^2}\Big)}{(t-t')^2 - |\mathbf{s}-\mathbf{q}(t')|^2}\Big(\mathbf{s} - \mathbf{q}(t') - \boldsymbol{v}(t')(t-t')\Big).
$$

The Volterra equation for the acceleration

The key proposition

Proposition (KTZ) *Given $C^{0,1}$ maps $t \mapsto \mathbf{q}(t)$ and $t \mapsto \mathbf{p}(t)$, with Lip($\mathbf{q}$) $= v$, Lip($\mathbf{v}$) $= a$, and $|\mathbf{v}(t)| \leq v < 1$, the Volterra equation as a fixed point map has a unique C^0 solution $t \mapsto \mathbf{a}(t) = \alpha[\mathbf{q}(\,\cdot\,), \mathbf{p}(\,\cdot\,)](t)$. Moreover, the solution depends Lipschitz continuously on the maps $t \mapsto \mathbf{q}(t)$ and $t \mapsto \mathbf{p}(t)$.*

The proof takes several dozen pages of careful estimates, but at the end of the day it all pans out! The well-posedness result for the joint initial value problem of MBLTP fields and their point charge sources is a corollary of the above Proposition.

Motion along a constant electric capacitor field

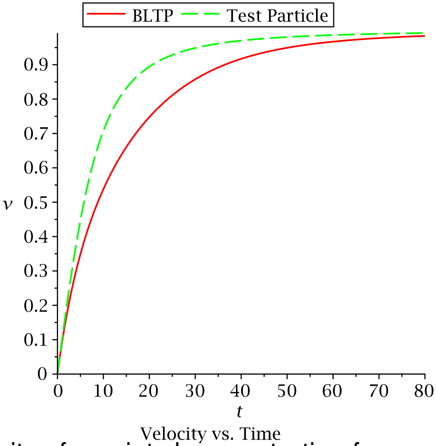

Figure: The velocity of a point charge, starting from rest in a strong, constant applied electrostatic field **E**.

Motion along a constant electric capacitor field

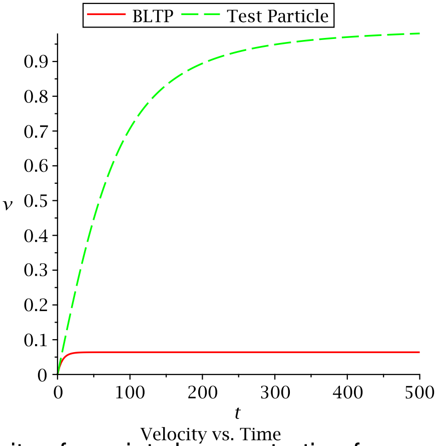

Figure: The velocity of a point charge, starting from rest in a very weak, constant applied electrostatic field **E**.

Motion along a constant electric capacitor field

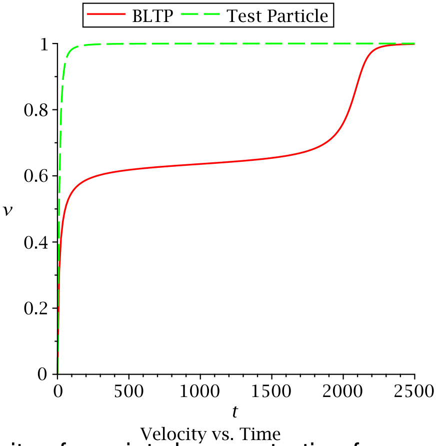

Figure: The velocity of a point charge, starting from rest in a constant applied electrostatic field **E** of slightly larger-than-critical field strength.

Does $G = 0$ electrodynamics have $G > 0$ neighborhood?

Generally covariant formulation of the Cauchy problem for Einstein gravity coupled to electromagnetism with point charge sources having bare mass requires:

- electromagnetic Maxwell equations are equipped with nonlinear or higher-order linear electromagnetic vacuum laws to guarantee integrable field-energy and -momentum densities and mild curvature singularities,
- no Black Holes! $\to$ Energy-momentum-stress tensor of each particle has negative (or zero) bare mass.
- weak 2^{nd} Bianchi identity (j/w Burtscher, Stalker, STZ)
- $\nabla \cdot \boldsymbol{T} = \mathbf{0} \implies$ well-defined equations of motion ($\leftarrow$ Big ?!)

No well-defined joint Cauchy problem yet!

The Genie is out of the bottle again!

Fin!

THANK YOU FOR LISTENING!

Preprints

Burtscher, A., Kiessling, M.K.-H., Stalker, J., and Tahvildar-Zadeh, A.S.,
Weak second Bianchi identity for spacetimes with timelike singularities,
20pp., submitted to Class. Quant. Grav. (2018)

Kiessling, M.K.-H.,
Force on a point charge source of the classical electromagnetic field,
Phys. Rev. D **100**, art.065912(19pp) (2019).

Kiessling, M.K.-H., and Tahvildar-Zadeh, A.S.,
Bopp–Landé–Thomas–Podolsky electrodynamics as initial value problem,
Rutgers Univ. Preprint, 95+pp. (in preparation, 2019)