Math Overboard

The Art of the Long Race

Art Benjamin

So, what are your chances of winning the race? This is a question that you probably ask yourself in almost every game you play, perhaps several times per game. We want to know this information to help us determine appropriate cube actions, as well as when choosing between different game plans.

In this article, I will focus on long races, where both players have more than 60 pips to go. These are the easiest races to evaluate. We will look at shorter races in future articles.

Let's begin with cube action. Although many authors have weighed in on this subject, I think Walter Trice had the right idea. (I'll note that my results and formulae differ slightly from the way Walter described it, but all discrepancies have been verified by XG.) Let X denote the pip count of the player on roll (whom we will call the "leader" even though that may not always be the case) and let Y denote the pip count of the other player, whom we will call the "trailer." Assuming that X and Y are greater than 60, we compute the Point of Last Take (PLT) by adding 10 percent of the leader's pip count, rounding down, then adding 2. Algebraically,

$$\text{PLT} = X + \lfloor X/10 \rfloor + 2$$

where the floor symbol $\lfloor \square \rfloor$ indicates that the quantity should be rounded down to the nearest integer. For example, $\lfloor 8.6 \rfloor = 8$. (Notice that this formula does not require any division. The floor of X/10 is just the number X with the last digit removed.) Thus, if the leader has X = 86 pips, then PLT = 86 + 8 + 2 = 96. Thus, if Y ≤ 96, the trailer can take. If Y > 96, the trailer should pass. When should the leader double? According to Trice, the leader should double when the trailer is 3 pips below PLT and redoubles when the trailer is 2 pips below PLT. That is, the Doubling Point DP and the Redoubling Point RDP satisfy

$$\text{DP} = \text{PLT} - 3$$
$$\text{RDP} = \text{PLT} - 2$$

Thus, if the leader has 86 pips, then since PLT = 96, the leader should double when Y ≥ 93, and should redouble when Y ≥ 94. Other racing formulas, such as the Keith Count and the Isight count define the doubling point to be DP = PLT – 4 (and RDP = PLT – 3). Which is correct? After further testing, I have concluded that when Y = PLT – 4, it is right on the border of Double/NoDouble, so either decision would be less than a 0.02 (and probably less than 0.01) error, so the double is optional. But when Y = PLT – 3, the doubling decision is not borderline, you must double for money and at money-like match scores.

Let's do one more example. If the leader has X = 95 pips, then PLT = 95 + 9 + 2 = 106. The leader should double if Y ≥ 103, redouble if Y ≥ 104, and the trailer should take if Y ≤ 106. (The leader has an optional double at Y = 102 and an optional redouble at Y = 103.)

Perhaps an easier way to remember this rule is to say that for long races, when the leader has X pips, the doubling window is 10% ± 2. In other words, the Double/Take region is when

$$X+\lfloor X/10\rfloor-2\leq Y\leq X+\lfloor X/10\rfloor+2$$

keeping in mind that the bottom number is an optional double. For those, more visually inclined, see Figure 1 below.

Figure 1. For a long race, the leader has an optional double (OD) with a lead of 10% – 2, an initial double (DP) with a lead of 10% –1, a redouble (RDP) with a 10% lead, and the trailer's Point of Last Take (PLT) is with a lead of 10% + 2.

These doubling rules are accurate for almost all values of X above 60 and below 130 (and I can't remember ever needing to determine PLT for a race above 130). The exceptions are handled by the "9 Rule." If X = 69, 79, 89, or 99, then add one more pip to the previous formula. These pip counts have PLT = 78, 89, 100, and 111, respectively. (They aspire to be one higher than their first digit suggests.)

Caveat: The doubling rules given above are highly accurate when both players have "low wastage" positions, which means that there are very few checkers on the 3 point or below. When checkers are on the low points, and especially when a low point has three or more checkers, then we should adjust the pip counts to reflect the extra wastage. As I mentioned in my Fall 2019 column on Pip Counting, I count the third and higher checker on the ace point as 3 pips, the third and higher checker on the deuce point as 3 pips, and the third and higher checker on the 3 point as 3.5 pips. When checkers begin to accumulate on low points, especially for the trailer, an adjustment to the pip count may be necessary for improved accuracy by taking checkers off and crossovers into account.

Although this article is about long races, let me briefly note the situation for shorter races. When the leader has a pip count X that is 60 or below, then as described by Trice, the Point of Last Take is determined by

$$PLT= X+ \lfloor (X-5)/7\rfloor$$

The doubling and redoubling points satisfy the same rules as before, namely DP = PLT – 3, and RDP = PLT – 2. Again, these rules should only be applied to low wastage positions, where nearly all of the inner board checkers are on the high points. When this is not the case, then there are other variables besides the pip count in play, and that will be the subject of a future article.

We now return to our original question. Given the pip counts, what are your chances of winning the race? The first reasonable answer to this question was given by Danny Kleinman in 1980 in his book, *Vision Laughs at Counting with Advice to the Dicelorn*. Using properties of the normal distribution, he arrives at the following formula. As before, let X be the leader's pip count (on roll), let Y be the trailer's pip count, and define D = (Y – X) + 4. (The +4 is for being on roll, noting that the average dice roll is slightly bigger than 8.) Then we compute the quantity K (as in Kleinman) as follows:

$$K=D^2/(X+Y-4)$$

The value of K leads to a good approximation of the doubler's cubeless game winning chance (GWC) by way of a table, for which the most important values are shown in Table 1.

Table 1. Kleinman's estimate of Game Winning Chances

K	0.5	0.7	1.0	1.2	1.4	2.0
GWC	69	72	76	78	80	84

For example, in a race where X = 70 and Y = 80, then D = 14, so K = 142/(70 + 80 – 4) = 196/146 = 1.34, which suggests that the leader will win a little more than 79% of the time (and thus trailer wins less than 21% of the time) suggesting that this would be a close pass for money. This is consistent with our PLT formula, since when the leader has 70 pips, the trailer can take with 79 pips or less. Although reasonably accurate, K can be difficult to calculate over the board.

In 2007, I collaborated with two former students of mine on the paper Estimating Winning Probabilities in Backgammon Races, where we used slightly more sophisticated mathematical tools and simulation of real backgammon positions to arrive at a more accurate estimate for GWC. Here we compute the statistic R (which I am naming after the lead author, Andrew Ross, now a professor of mathematics and statistics at Eastern Michigan University) as follows. As before, we let X be the leader's pip count, Y be the trailer's pip count, and D = (Y – X) + 4, then

$$R=(D^2+D/7)/(X+Y-25)$$

You then look up the number R in Table 2, which produces the Game Winning Chances.

If we let X = 70 and Y = 80, as in our previous example, then D = 14 and R = (14^2 + 14/7)/(70 + 80 – 25) = 198/125 = 1.584, which predicts that the leader will win slightly more than 80% of the time, which is a little more accurate.

Table 2. The Ross-Benjamin table. Please don't memorize it.

R	GWC		R	GWC
0.000	0.50		0.998	0.75
0.001	0.51		1.095	0.76
0.006	0.52		1.198	0.77
0.012	0.53		1.308	0.78
0.022	0.54		1.427	0.79
0.035	0.55		1.554	0.80
0.050	0.56		1.691	0.81
0.068	0.57		1.838	0.82
0.089	0.58		1.997	0.83
0.114	0.59		2.170	0.84
0.140	0.60		2.357	0.85
0.171	0.61		2.561	0.86
0.204	0.62		2.784	0.87
0.242	0.63		3.029	0.88
0.282	0.64		3.301	0.89
0.326	0.65		3.603	0.90
0.373	0.66		3.944	0.91
0.425	0.67		4.331	0.92
0.480	0.68		4.778	0.93
0.539	0.69		5.304	0.94
0.603	0.70		5.936	0.95
0.672	0.71		6.724	0.96
0.745	0.72		7.761	0.97
0.824	0.73		9.254	0.98
0.908	0.74		11.874	0.99

Unfortunately, since R is at least as difficult to compute over the board as K, I decided that the statistic was too much work for practical use, and put it aside for a while. But over the years, I heard that top players (including Jacob "Stick" Rice, Petko Kostadinov, and especially Grant Hoffman) were using this table and finding it to be extremely accurate, and was consistent with XG calculations.

(Grant calls it the RB count to denote Ross-Benjamin or Racing Benchmark.) I was pleased to know that this method was so accurate, but troubled that it was so difficult to calculate. Perhaps there was a way to harness the power of the RB count, without having to do any messy calculations or memorize too much? Happily, the answer turns out to be yes.

Let's start by examining the game winning chances when the race is very close, but not close enough to be considered a money cube. As most players know, if you are on roll and trailing by exactly 4 pips, then your GWC = 50 percent. For example, if the player on roll has 88 pips, and their opponent has 84 pips, then the race is even. This makes sense, since the average roll in backgammon is about 8, so after trailing by 4 pips and rolling an 8, you will be leading by 4 pips, and you put your opponent in the same situation that you were in. In other words, expressing GWC in percentages,

$$\text{If X} = \text{Y} + 4\text{, then GWC} = 50$$

What happens when the pip count is exactly even? Using the RB table, you see that when Y = X = 100, then D = 4, and R = 16.57/175 = 0.095, indicating that GWC = 58%. Comparing this to the 50% figure when trailing by 4, we see that each pip is worth about 2%. When Y = X = 61, then R = 16.57/95 = 0.171, indicates a winning chance of about 61% (easy to remember!). Comparing this to the 50% figure, we see that each pip is worth about 2.75% in that region. Rather than memorize exact formulas for situations that rarely need such precision, I summarize everything into a simple approximation. For long close races, where X and Y are greater than 60, let Y – X = Δ denote the lead of player on roll (which could be negative). If -5 ≤ Δ ≤ 5, then the roller's game winning chance is

$$\text{GWC} \approx 60 + 2\Delta$$

This approximation should almost always be within 2% of the exact game winning chance. For example, if the player on roll has X = 75 pips and the trailer has Y = 78 pips, then the player on roll has a lead of Δ = 3, and thus has winning chances approximately equal to 60 + 6 = 66 percent. (The RB formula says D = 7 and R = 50/128 = 0.39, suggesting that GWC is about 66.3%.)

The above formula gives an excellent approximation for the game winning chances in close races, when the leader's game winning chance is between 50 and 70 percent. When the leader's game winning chance is above 70%, it is important to have a more accurate approximation, since this is the region where most cube decisions will be made. Happily, the PLT formula provided earlier gives us another useful benchmark. In a long race, when the trailer is at the point of last take (for money), the leader's winning chance is always very close to 78% (usually within about half a percent). In other words, for X, Y > 60,

$$\text{If Y} = \text{PLT, then GWC} = 78.$$

Equivalently, at the point of last take, the Trailer's Winning Chance (TWC) is about 22%. That is, for X, Y > 60,

$$\text{If Y} = \text{PLT, then TWC} = 22.$$

Earlier, we saw that when GWC is in the 50% to 70% range, each pip is worth about 2%. But when we are within 5 pips of the PLT, each extra pip is worth about 1.5%. In other words, for X, Y > 60, and for –5 ≤ e ≤ 5,

$$\text{If Y} = \text{PLT} + \text{e,}$$
$$\text{then TWC} = 22 - 1.5\text{e}$$

For a little more accuracy, if the trailer is e pips *beyond* PLT, then TWC = 22 – 1.4e, and if the trailer is e pips *below* PLT, then TWC = 22 + 1.6e, but the above formula is accurate enough.

Table 3 gives the Trailer's Winning Chances relative to their point of last take, where the numbers are rounded to the nearest half-percent. Conveniently, the endpoints of the interval are nice round numbers: when you are 5 pips beyond the PLT, your winning chances are 15%, when you are 5 pips below the PLT, your winning chances are 30%.

Table 3. The Trailer's cubeless Winning Chances (TWC) relative to the point of last take, rounded to the nearest half-percent.

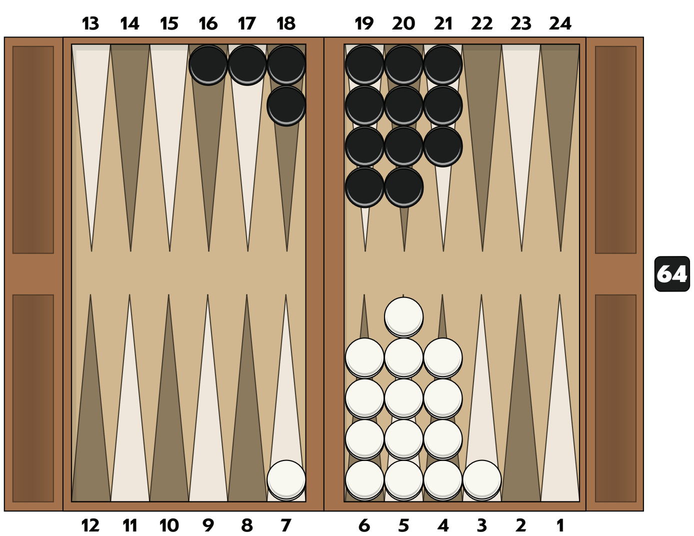
Table 3. Kleinman's estimate of Game Winning Chances

Y – PLT	–5	–4	–3	–2	–1	0	+1	+2	+3	+4	+5
TWC	30	28.5	27	25	23.5	22	20.5	19	18	16.5	15

Another way to remember Table 3, is to always increase or decrease the GWC by 1.5% except when you go from the 2 level to the 3 level. That is, when we go from +2 to +3, the trailer's GWC decreases by 1%; when we go from –2 to –3, the trailer's GWC increases by 2%.

Let's do a quick example. Consider the position below. White has 75 pips and Red has 87 pips, with White on roll. Suppose White is leading 5 to 3 in a 7-point match, so Red has a take point of 19% (or 18.5% if you want extra accuracy). What is the proper cube action?

White on Roll. What are Black's Game Winning Chances?



Score		Pips
4-Away	White	87
2-Away	Black	75

Since there are very few checkers on the lower points, this constitutes a long, low-wastage racing position, so crossovers won't matter. By the Trice formula, PLT = 75 + 7 + 2 = 84, so the Trailer can take for money with as many as 84 pips. Since the trailer has 3 pips more than that, then our table indicates that their winning chances are about 18%, which is below the 19% take point. Thus, White should double and Red should pass. XGR++ confirms these numbers, saying that Red wins 17.95% of the time, and that taking would be a 0.059 error. (The RB formula also confirms this. Here, R = 258.29/137 = 1.88, suggesting that the Leader's GWC is slightly more than 82%.)

Next comes my favorite part of the table. when the trailer is 5 to 10 pips beyond the PLT for money, the trailer's win probability decreases by 1 percent per pip, as shown in Table 4.

Table 4: When Trailer is 5 to 10 pips beyond PLT, then apply the Rule of 20.

Y – PLT	5	6	7	8	9	10
TWC	15	14	13	12	11	10

This leads to the Rule of 20: Suppose the trailer is 5 to 10 pips beyond PLT. That is, Y = PLT + e, where 5 ≤ e ≤ 10. Then the Trailer's cubeless Winning Chances are

$$\text{TWC} = 20 - e.$$

For example, if X = 75, and Y = 92, then PLT = 84, so the trailer is 8 pips beyond PLT, suggesting that TWC = 12%. In the previous position, if we move a checker from Red's 5 point to their 10 point, we create such a situation and XGR++ indicates that TWC = 12.00%.

When the trailer is more than 10 pips beyond PLT, their game winning chances drop by about 0.6% per pip, so that at 20 pips beyond PLT, the trailer has just 4% game winning chances, as shown in Table 5, with numbers rounded to the nearest half-percent.

Table 5. When Trailer is 10 to 20 pips beyond PLT, then apply the Rule of 10.

Y – PLT	10	11	12	13	14	15	16	17	18	19	20
TWC	10	9.5	9	8	7.5	7	6.5	6	5	4.5	4

The formula I use here is for X, Y > 60 and for $0 \le e \le 10$, suppose Y = PLT + 10 + e, then the trailer's winning chances are given by the Rule of 10:

$$\text{TWC} = 10 - 0.6\text{e}$$

Alternatively, you could just start at 10% and count down by 0.5%, remembering to drop a full 1% on the third step and on the eighth step.

With these simple tools at your disposal, you are now in a position to estimate game winning chances (between 4% and 96%) for most long low-wastage races that arise in practice.

Acknowledgments: I am extremely grateful to David Presser, John O'Hagan, Ilia Guzei, Tom Courts, Daphne Downs, and Marty Storer for valuable ideas and feedback.

References:

Art Benjamin, Math Overboard: Handy Tips for Counting Pips, *PrimeTime Backgammon*, Fall 2019, pp. 86-93.

Danny Kleinman, Normal Race Takes, *Vision Laughs at Counting with Advice to the Dicelorn*, 1980, pp. 142-143 (reprinted by Lulu Press, 2007).

Andrew M. Ross, Arthur Benjamin, and Michael Munson, Estimating Winning Probabilities in Backgammon Races, in O*ptimal Play: Mathematical Studies of Games and Gambling*, University of Nevada, Reno, 2007, pp. 269-291.

Walter Trice, *Backgammon Boot Camp*, Fortuitous Press, San Francisco, 2004.

Tournament Recap

Mind Sports Olympiad, August 2020

Cristian Frisk

The 24th Annual Mind Sports Olympiad (MSO) ended on Sunday, August 30, after a month of hotly contested online competition ranging from Scrabble and Speed Reading to Chess, Backgammon, and Catan. There were 100 events, which included eight backgammon events, and eight meta-events where the scores of individual events were combined. One hundred and six countries took part, and 56 countries won medals. Several countries participated for the first time, including Kyrgyzstan, Bolivia, Guatemala, Malawi, Moldova, Mozambique, Tanzania, Uzbekistan, and Zimbabwe.

The medalists for backgammon events are:

Backgammon (11 points)

1. Michael Mesich (USA)
2. Peter Wisniewski (USA)
3. Alesia Mercuri (Italy) and Daniela Tunsoiu (Romania)

Backgammon (Double Match Point)

1. Arad Pedram (Iran)
2. Julia Hayward (England)
3. Piero Zama (Italy) and Robert Kreisl (Austria)

Hypergammon (3 points)

1. Robert Kreisl (Austria)
2. David Pearce (England)
3. Scott Agius and Simon Jones (both from England)