Identity Found by Proving Identities

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Experimental Mathematics Seminar
Rutgers University

FPSAC

Early Days of FPSAC

3rd edition of FPSAC 1991 in Bordeaux:

- 5 invited talks

FPSAC 1991 : Bordeaux (France)

- Pierre Cartier: Paris, France
- Philippe Flajolet: Rocquencourt, France
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- highlight: thesis defense of Mireille Bousquet-Mélou

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Doron Zeilberger's Invited Talk 1991

IDENTITIES IN SEARCH OF IDENTITY

Doron ZEILBERGER

Department of Mathematical Sciences

Drexel University

Philadelphia

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Abstract

The time is ripe to start a science of identities for their own sake, without paying lip-service to hight-brow mathematics. Although putting combinatorial, (abstract-) algebraic or analytic flesh and blood on identities did lead and will lead to considerable insight as well as new identities, there is also much to be gained in forgetting advanced mathematics, and starting a new sub-discipline of high-school mathematics called "the theory of identities".

Doron Zeilberger's Invited Talk 1991

Definition. A mathematical sentence that has "$=$" in its middle is called an *identity*.

The format of an identity is thus

$$\text{SOMETHING} = \text{SOMETHING ELSE}.$$

Trivial Example. $\mathrm{Re}(s) = \frac{1}{2}$, for every complex zero s of $\zeta(s)$.

Easy Example. ANALYTIC INDEX $=$ TOPOLOGICAL INDEX.

Deep Example. $1 + 1 = 2$.

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Definition: A term $f(n)$ is called hypergeometric if

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Remark: Generalize geometric sequences where $\dfrac{f(n+1)}{f(n)} = \text{const}.$

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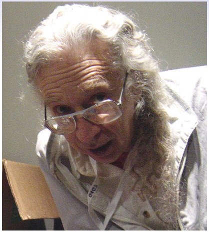
Remark: Generalize geometric sequences where $\frac{f(n+1)}{f(n)} = \text{const}.$

Examples: $\text{rat}(n)$, x^n, $n!$, $(a)_n$, $\binom{2n}{n}$, $\Gamma(3n+1)$, etc.

Gosper's algorithm



Proc. Natl. Acad. Sci. USA
Vol. 75, No. 1, pp. 40–42, January 1978
Mathematics

Decision procedure for indefinite hypergeometric summation

(algorithm/binomial coefficient identities/closed form/symbolic computation/linear recurrences)

R. WILLIAM GOSPER, JR.

Xerox Palo Alto Research Center, Palo Alto, California 94304

Communicated by Donald E. Knuth, September 26, 1977

ABSTRACT Given a summand a_n, we seek the "indefinite sum" $S(n)$ determined (within an additive constant) by

$$\sum_{n=1}^{m} a_n = S(m) - S(0) \quad [0]$$

or, equivalently, by

$$a_n = S(n) - S(n-1). \quad [1]$$

An algorithm is exhibited which, given a_n, finds those $S(n)$ with the property

$$\frac{S(n)}{S(n-1)} = \text{a rational function of } n. \quad [2]$$

erate case where a_n is identically zero.) Express this ratio as

$$\frac{a_n}{a_{n-1}} = \frac{p_n}{p_{n-1}} \frac{q_n}{r_n}, \quad [5]$$

where p_n, q_n, and r_n are polynomials in n subject to the following condition:

$$\gcd(q_n, r_{n+j}) = 1, \quad [6]$$

for all non-negative integers j.

It is always possible to put a rational function in this form, for if $\gcd(q_n, r_{n+j}) = g(n)$, then this common factor can be

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Examples:

- $$\sum_{k=0}^{n} \frac{(4k+1)\,k!}{(2k+1)!} = \sum_{k=0}^{n} \left(\frac{2k!}{(2k)!} - \frac{2(k+1)!}{(2k+2)!} \right) = 2 - \frac{n!}{(2n+1)!}$$

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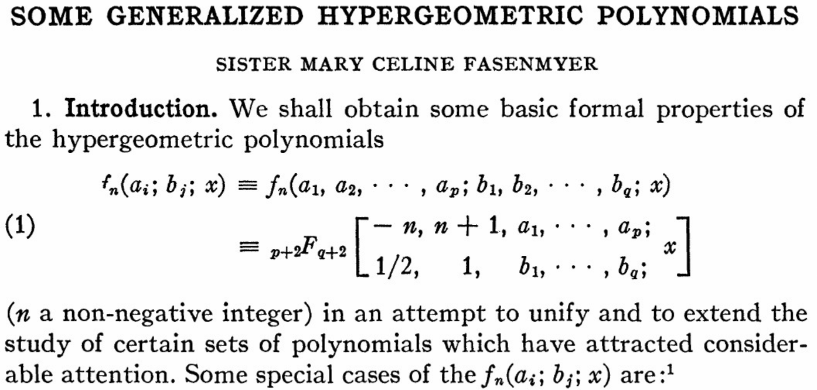
Question: What about definite hypergeometric summation

$$\sum_{k=0}^{n} f(n,k) = \,? \qquad \text{Such as } \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Fasenmyer's Algorithm

- aka "Sister Celine's algorithm"
- developed in her doctoral thesis in 1945



SOME GENERALIZED HYPERGEOMETRIC POLYNOMIALS

SISTER MARY CELINE FASENMYER

1. Introduction. We shall obtain some basic formal properties of the hypergeometric polynomials

$$f_n(a_i; b_j; x) \equiv f_n(a_1, a_2, \cdots, a_p; b_1, b_2, \cdots, b_q; x)$$

(1)
$$\equiv {}_{p+2}F_{q+2}\left[\begin{matrix} -n, & n+1, & a_1, \cdots, a_p; \\ 1/2, & 1, & b_1, \cdots, b_q; \end{matrix} \; x\right]$$

(n a non-negative integer) in an attempt to unify and to extend the study of certain sets of polynomials which have attracted considerable attention. Some special cases of the $f_n(a_i; b_j; x)$ are:[1]

Sister Celine's Algorithm

Algorithm: given hg. $f(n,k)$, find recurrence for $\sum_{k=-\infty}^{\infty} f(n,k)$.

1. Choose $r, s \in \mathbb{N}$ (order in n, order in k).
2. Ansatz for a k-free recurrence: $\sum_{i=0}^{r} \sum_{j=0}^{s} c_{i,j} \cdot f(n+i, k+j)$.
3. Divide by $f(n,k)$ and simplify.
4. Multiply by the common denominator.
5. Perform coefficient comparison with respect to k.
6. Solve the linear system for the $c_{i,j} \in \mathbb{K}(n)$.
7. Sum over the k-free recurrences and return the result.

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Example: $F(n) := \sum_{k=0}^{n} f(n,k) = \binom{2n}{n}$ with $f(n,k) := \binom{n}{k}^2$.

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With $r = s = 2$ we find the k-free recurrence

$$\begin{aligned} 0 = &-(n+1)f(n,k) + (2n+2)f(n,k+1) - (n+1)f(n,k+2) \\ &+ (2n+3)f(n+1,k+1) + (2n+3)f(n+1,k+2) - (n+2)f(n+2,k+2) \end{aligned}$$

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Collecting terms: $(4n+6)F(n+1) - (n+2)F(n+2) = 0$.

Wilf–Zeilberger (WZ) Theory

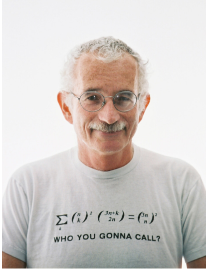
Invent. math. 108: 575–633 (1992)

Inventiones mathematicae

An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities

Herbert S. Wilf* and Doron Zeilberger**

Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104, USA
Department of Mathematics, Temple University, Philadelphia, PA 19122, USA



J. Symbolic Computation (1991) **11**, 195–204

The Method of Creative Telescoping

DORON ZEILBERGER

Department of Mathematics and Computer Science, Temple University, Philadelphia, PA 19122, *USA*

In memory of John Riordan, master of ars combinatorica

(*Received* 1 *June* 1989)

An algorithm for definite hypergeometric summation is given. It is based, in a non-obvious way, on Gosper's algorithm for definite hypergeometric summation, and its theoretical justification relies on Bernstein's theory of holonomic systems.

Zeilberger's Fast Algorithm

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(we assume natural boundaries, e.g., f has finite support w.r.t. k).

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- Try order $r = 0, 1, 2, \ldots$ until success.

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But one does not know it, neither its order nor its coefficients.

- Try order $r = 0, 1, 2, \ldots$ until success.
- Write recurrence with undetermined coefficients $p_i \in \mathbb{K}(n)$:

$$p_r(n)F(n + r) + \cdots + p_1(n)F(n + 1) + p_0(n)F(n) = 0.$$

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- Apply a parametrized version of Gosper's algorithm to

$$p_r(n)f(n+r,k) + \cdots + p_1(n)f(n+1,k) + p_0(n)f(n,k).$$

Creative Telescoping

Method for doing integrals and sums
(aka Feynman's differentiating under the integral sign)

Consider the following summation problem: $F(n) := \sum_{k=a}^{b} f(n,k)$

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Summing from a to b yields a recurrence for $F(n)$:

$$c_r(n) F(n+r) + \cdots + c_0(n) F(n) = g(n,b+1) - g(n,a).$$

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Consider the following integration problem: $F(x) := \int_a^b f(x,y)\,\mathrm{d}y$

Telescoping: write $f(x,y) = \frac{\mathrm{d}}{\mathrm{d}y} g(x,y)$.

Then $F(n) = \int_a^b \left(\frac{\mathrm{d}}{\mathrm{d}y} g(x,y)\right)\mathrm{d}y \qquad = g(x,b) - g(x,a)$.

Creative Telescoping: write

$$c_r(x)\frac{\mathrm{d}^r}{\mathrm{d}x^r} f(x,y) + \cdots + c_0(x) f(x,y) \ = \frac{\mathrm{d}}{\mathrm{d}y} g(x,y).$$

Integrating from a to b yields a differential equation for $F(x)$:

$$c_r(x)\frac{\mathrm{d}^r}{\mathrm{d}x^r} F(x) + \cdots + c_0(x) F(x) \ = g(x,b) - g(x,a)$$

Beyond Hypergeometric: Holonomic Functions

Definition: A sequence $f(n)$ is called **P-recursive** if it satisfies a linear recurrence equation with polynomial coefficients:

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- Equivalently, such functions/sequences are called **holonomic**.
- Generalizations to several variables and mixed cases exist.
- In any case, one needs only finitely many initial conditions.

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$$p_r(x)f^{(r)}(x)+\cdots+p_1(x)f'(x)+p_0(x)f(x)=0 \qquad (p_r \neq 0).$$

Remarks:

- Equivalently, such functions/sequences are called **holonomic**.
- Generalizations to several variables and mixed cases exist.
- In any case, one needs only finitely many initial conditions.
- The holonomic (finite!) data structure consists of a system of linear functional equations together with initial values.

Special Functions

- arise in physics (real-world) and mathematical analysis

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Airy function

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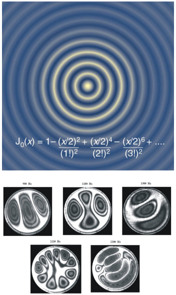

Bessel function

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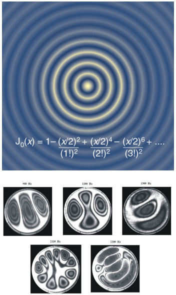

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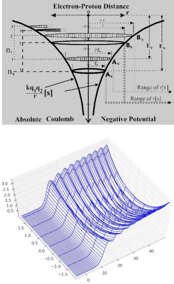

Coulomb function

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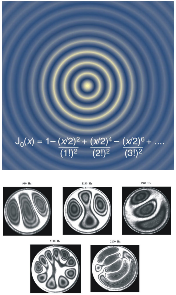

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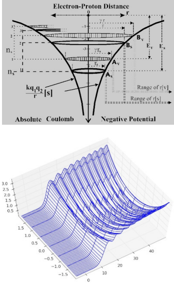

Coulomb function

Special Functions

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- are solutions to certain differential equations / recurrences
- cannot be expressed in terms of the usual elementary functions ($\sqrt{\ }$, exp, log, sin, cos, ...)

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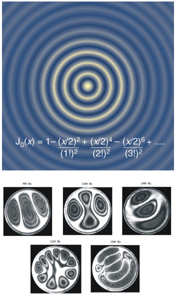

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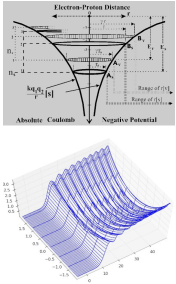


Coulomb function

The Holonomic Systems Approach

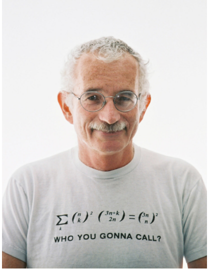
Journal of Computational and Applied Mathematics 32 (1990) 321–368 321
North-Holland

A holonomic systems approach to special functions identities *

Doron ZEILBERGER
Department of Mathematics, Temple University, Philadelphia, PA 19122, USA

Received 14 November 1989

Abstract: We observe that many special functions are solutions of so-called holonomic systems. Bernstein's deep theory of holonomic systems is then invoked to show that any identity involving sums and integrals of products of these special functions can be verified in a finite number of steps. This is partially substantiated by an algorithm that proves terminating hypergeometric series identities, and that is given both in English and in MAPLE.



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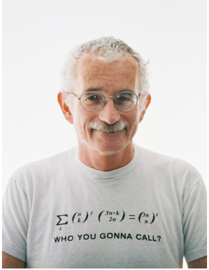

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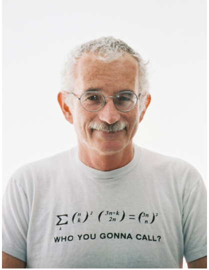

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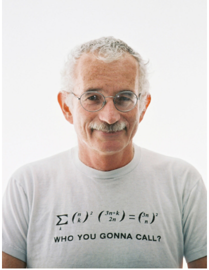

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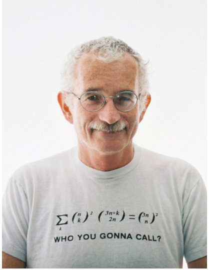

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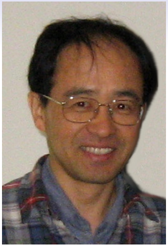
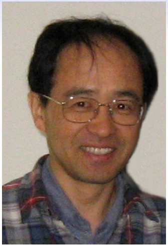


Takayama's Algorithm

An algorithm of constructing the integral of a module
— an infinite dimensional analog of Gröbner basis

NOBUKI TAKAYAMA

Department of Mathematics, Kobe University
Rokko, Kobe, 657, Japan



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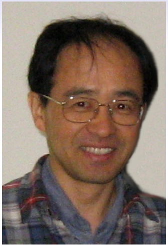
Recall: creative telescoping requires a relation of the form

- $c_r(n)f(n+r,k) + \cdots + c_0(n)f(n,k) = g(n,k+1) - g(n,k)$,
- or $c_r(x)\frac{\mathrm{d}^r}{\mathrm{d}x^r}f(x,y) + \cdots + c_0(x)f(x,y) = \frac{\mathrm{d}}{\mathrm{d}y}g(x,y)$.
- Left-hand side is called **telescoper**, g is called **certificate**.



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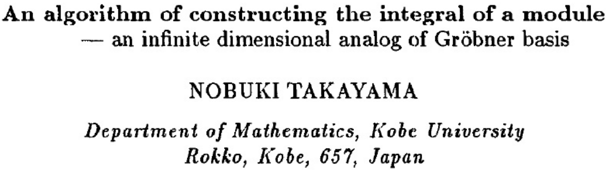

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Ideas of the Algorithm:

- Work in the setting of Weyl algebra and D-modules.
- It is not necessary to eliminate k (resp. y) completely.
- Note that the certificate g is not needed in certain situations.
- Based on elimination, uses Gröbner bases over modules.

Chyzak's Algorithm

Discrete Mathematics 217 (2000) 115–134

DISCRETE MATHEMATICS

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An extension of Zeilberger's fast algorithm to general holonomic functions☆

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- Solve it by uncoupling or by a direct method.
- Variation: C.K. proposed a heuristic approach that avoids the expensive uncoupling step (caveat: may not terminate).

Reduction-Based Creative Telescoping

Motivation:

- Typically, the certificate is much larger than the telescoper.
- Often it is not needed (natural boundaries / closed contour).
- Compute the telescoper without computing the certificate.

Contributors: Alin Bostan, Hadrian Brochet, Shaoshi Chen, Frédéric Chyzak, Hao Du, Lixin Du, Louis Dumont, Hui Huang, Manuel Kauers, Christoph Koutschan, Pierre Lairez, Ziming Li, Bruno Salvy, Michael Singer, Joris van der Hoeven, Mark van Hoeij, Rong-Hua Wang, Guoce Xin, . . .

Active Research Area: Google Scholar lists more than 1000 articles about creative telescoping.

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Reduction procedure (differential case): define $\rho\colon \mathcal{F} \to \mathcal{F}$ s.t.

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To compute a telescoper for $\int_a^b f(x,y)\,\mathrm{d}y$, apply the reduction ρ to the successive derivatives of the integrand f:

$$\begin{aligned} f &= g_0' + \rho\big(f\big) &&= g_0' + h_0, \\ \tfrac{\mathrm{d}}{\mathrm{d}x} f &= g_1' + \rho\big(\tfrac{\mathrm{d}}{\mathrm{d}x} f\big) &&= g_1' + h_1, \\ \tfrac{\mathrm{d}^2}{\mathrm{d}x^2} f &= g_2' + \rho\big(\tfrac{\mathrm{d}^2}{\mathrm{d}x^2} f\big) &&= g_2' + h_2,\ \dots \end{aligned}$$

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If the h_i live in a finite-dimensional $\mathbb{K}(x)$-vector space, then there exists a nontrivial linear combination $p_0 h_0 + \cdots + p_r h_r = 0$.

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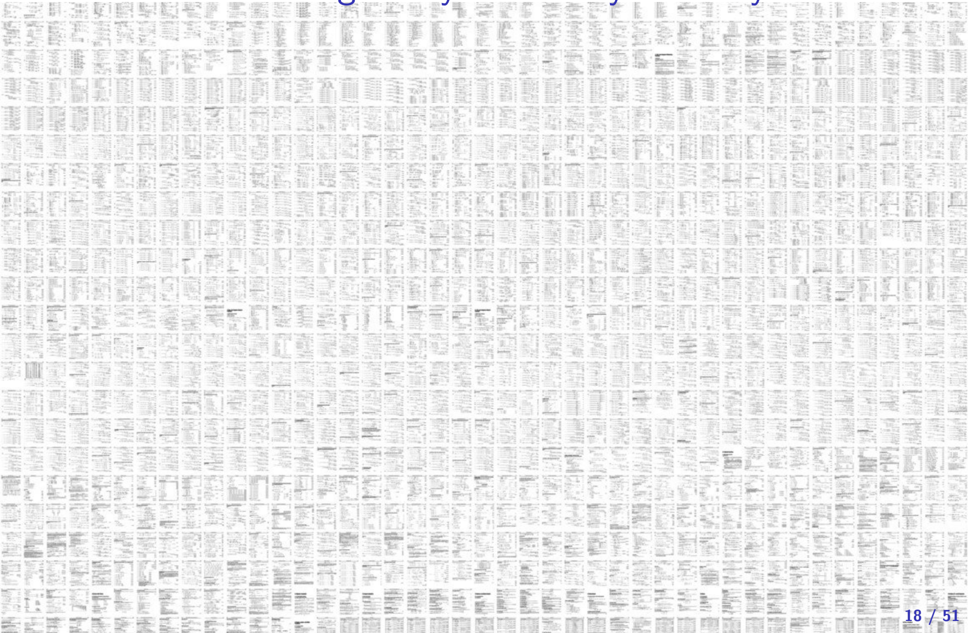
$\longrightarrow$ Hence, the desired telescoper is $p_0 f + p_1 f' + \cdots + p_r f^{(r)}$.

Table of Integrals by Gradshteyn and Ryzhik



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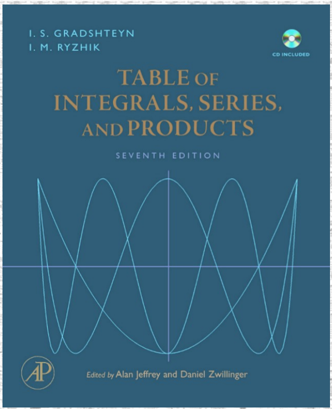

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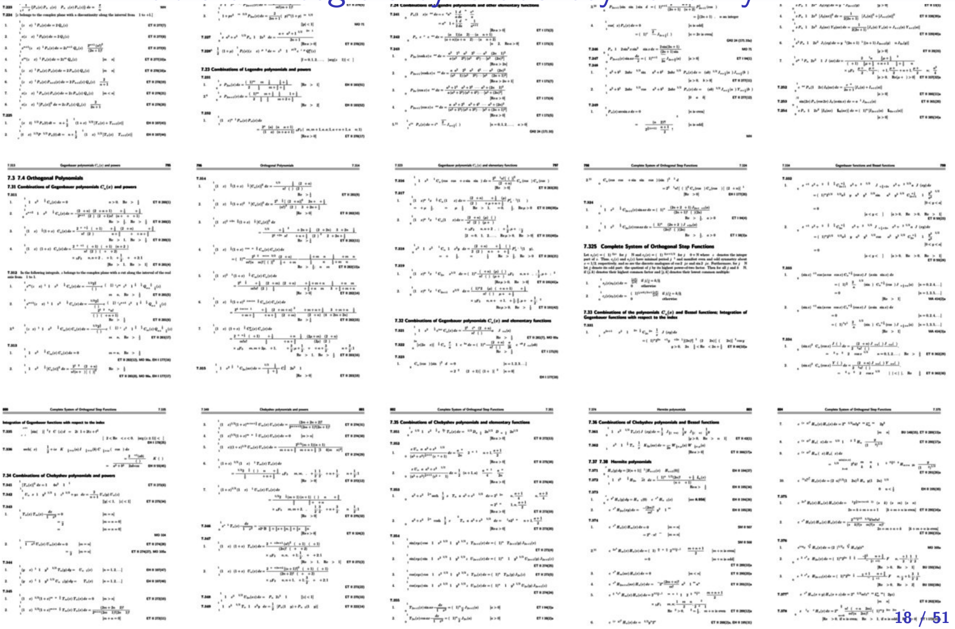

7.319

1. $$\int_0^1 (1-x)^{\mu-1} x^{\nu-1}\, C_{2n}^{\lambda}\left(\gamma x^{1/2}\right) dx = (-1)^n \frac{\Gamma(\lambda+n)\,\Gamma(\mu)\,\Gamma(\nu)}{n!\,\Gamma(\lambda)\,\Gamma(\mu+\nu)}\; {}_3F_2\left(-n, n+\lambda, \nu; \frac{1}{2}, \mu+\nu; \gamma^2\right)$$

$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ ET II 191(41)a

2. $$\int_0^1 (1-x)^{\mu-1} x^{\nu-1}\, C_{2n+1}^{\lambda}\left(\gamma x^{1/2}\right) dx = \frac{(-1)^n 2\gamma\,\Gamma(\mu)\,\Gamma(\lambda+n+1)\,\Gamma\left(\nu+\frac{1}{2}\right)}{n!\,\Gamma(\lambda)\,\Gamma\left(\mu+\nu+\frac{1}{2}\right)} \times {}_3F_2\left(-n, n+\lambda+1, \nu+\tfrac{1}{2}; \tfrac{3}{2}, \mu+\nu+\frac{1}{2}; \gamma^2\right)$$

$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}]$ ET II 191(42)

7.32 Combinations of Gegenbauer polynomials $C_n^{\nu}(x)$ and elementary functions

7.321 $$\int_{-1}^1 \left(1-x^2\right)^{\nu-\frac{1}{2}} e^{iax}\, C_n^{\nu}(x)\, dx = \frac{\pi 2^{1-\nu} i^n\, \Gamma(2\nu+n)}{n!\,\Gamma(\nu)} a^{-\nu} J_{\nu+n}(a)$$

$[\operatorname{Re}\nu > -\frac{1}{2}]$ ET II 281(7), MO 99a

7.322 $$\int_0^{2a} [x(2a-x)]^{\nu-\frac{1}{2}}\, C_n^{\nu}\left(\frac{x}{a}-1\right) e^{-bx}\, dx = (-1)^n \frac{\pi\,\Gamma(2\nu+n)}{n!\,\Gamma(\nu)} \left(\frac{a}{2b}\right)^{\nu} e^{-ab}\, I_{\nu+n}(ab)$$

$[\operatorname{Re}\nu > -\frac{1}{2}]$ ET I 171(9)

7.323

1. $$\int_0^{\pi} C_n^{\nu}(\cos\varphi)\,(\sin\varphi)^{2\nu}\, d\varphi = 0 \qquad [n = 1, 2, 3, \ldots]$$

Table of Integrals by Gradshteyn and Ryzhik

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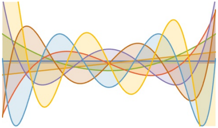

Gegenbauer polynomials $C_n^{(\alpha)}(x)$

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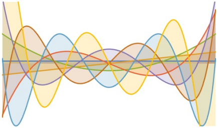

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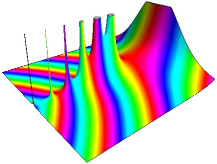

Gamma function $\Gamma(x)$

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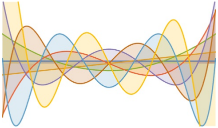

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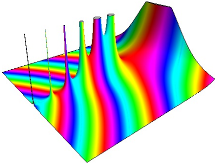

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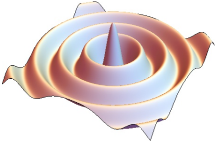

Bessel function $J_\nu(x)$

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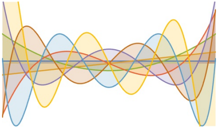

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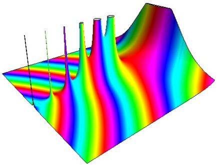

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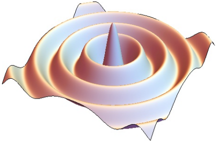

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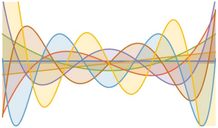
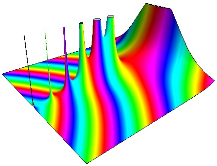
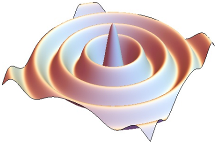
$$\int_{-1}^{1} \left(1-x^2\right)^{\nu-\frac{1}{2}} e^{iax}\, C_n^{\nu}(x)\, dx = \frac{\pi 2^{1-\nu} i^n\, \Gamma(2\nu+n)}{n!\, \Gamma(\nu)} a^{-\nu}\, J_{\nu+n}(a)$$

- A large portion of such identities can be proven via the holonomic systems approach.

Table of Integrals by Gradshteyn and Ryzhik







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- A large portion of such identities can be proven via the holonomic systems approach.
- Algorithms are implemented in the HolonomicFunctions package.

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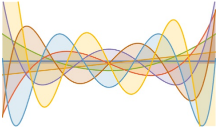

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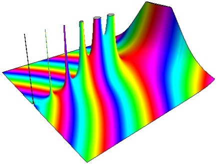

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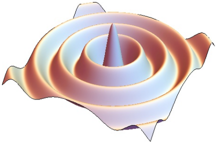

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Holonomic system, satisfied by both sides of the identity:

$$ia(n+2\nu)f_n'(a) + a(n+1)f_{n+1}(a) - in(n+2\nu)f_n(a) = 0,$$
$$a(n+1)(n+2)f_{n+2}(a) - 2i(n+1)(n+\nu+1)(n+2\nu+1)f_{n+1}(a)$$
$$-a(n+2\nu)(n+2\nu+1)f_n(a) = 0.$$

Random Walk Generating Functions

Study random walks on a lattice:

- d-dimensional integer lattice, or other
- certain set of allowed steps
- with or without restriction (positive quadrant or the like)
- univariate g.f. for excursions
- multivariate g.f. for walks with arbitrary endpoint

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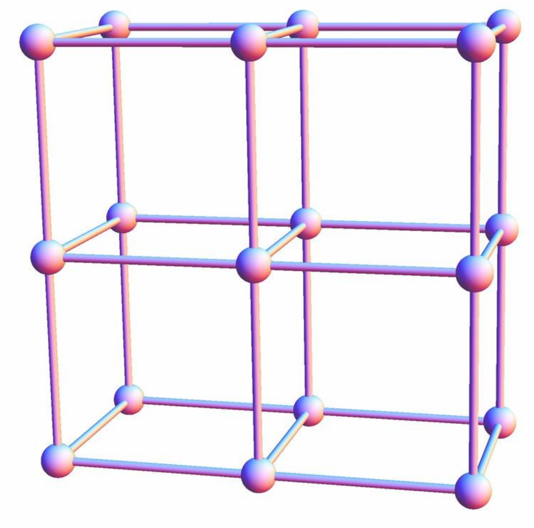
Some Contributors: Axel Bacher, Olivier Bernardi, Alin Bostan, Mireille Bousquet-Mélou, Manfred Buchacher, Frédéric Chyzak, Julien Courtiel, Guy Fayolle, Éric Fusy, Anthony Guttmann, Manuel Kauers, Irina Kurkova, Jean-Marie Maillard, Stephen Melczer, Marni Mishna, Kilian Raschel, Andrew Rechnitzer, Bruno Salvy, Gilles Schaeffer, Amélie Trotignon, Michael Wallner, . . .

Face-centered cubic (fcc) lattice

Example: Construction in 3D



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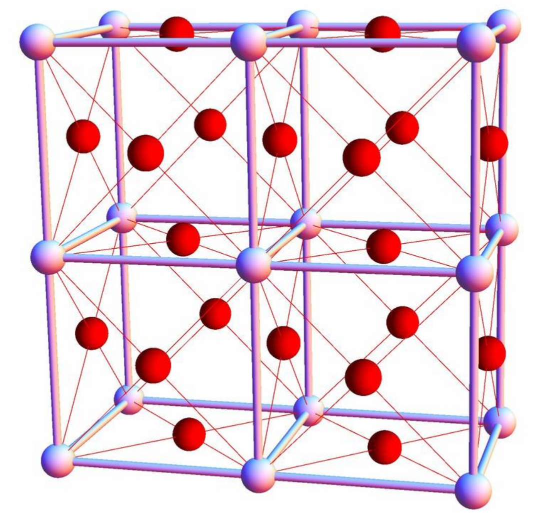

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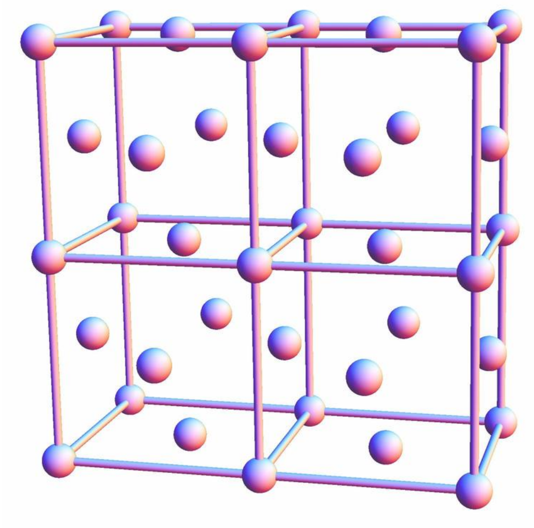

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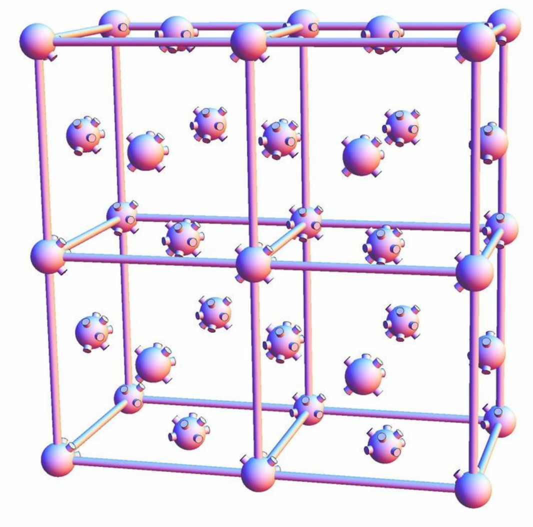

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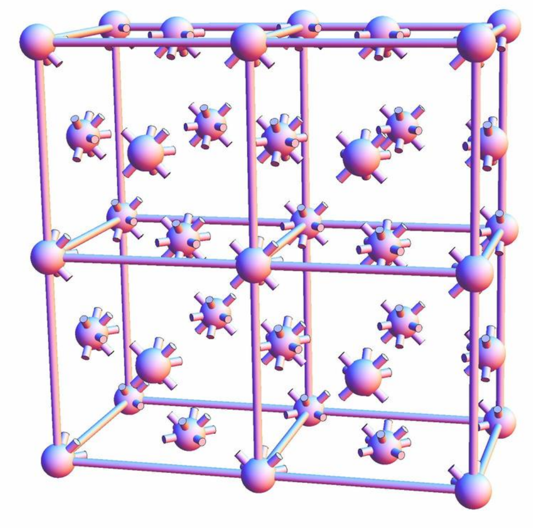

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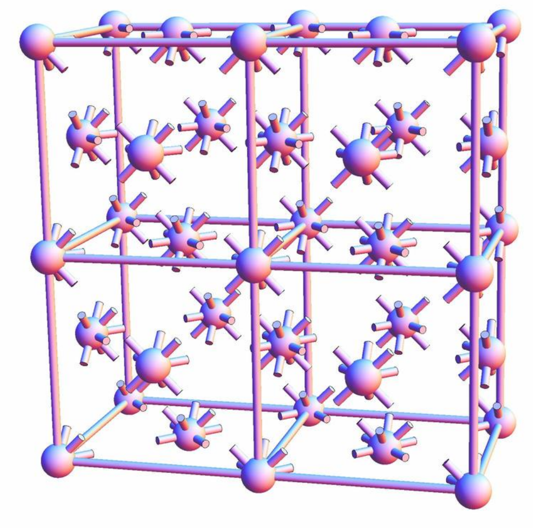

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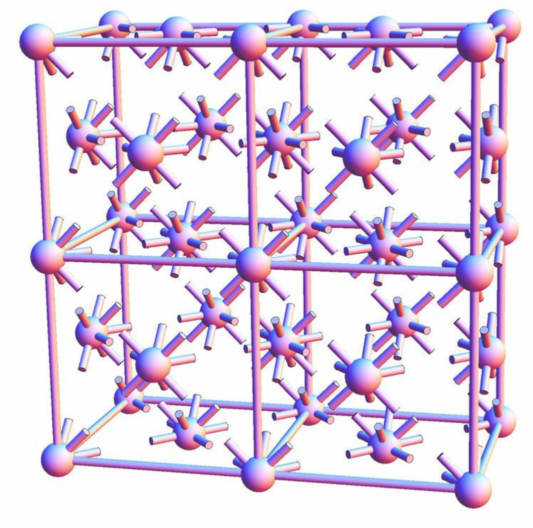

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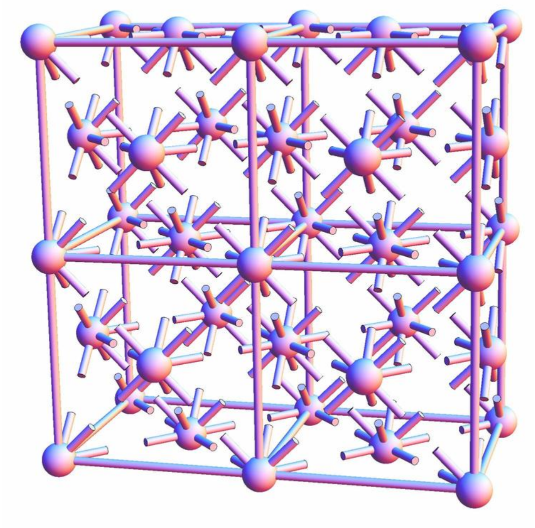

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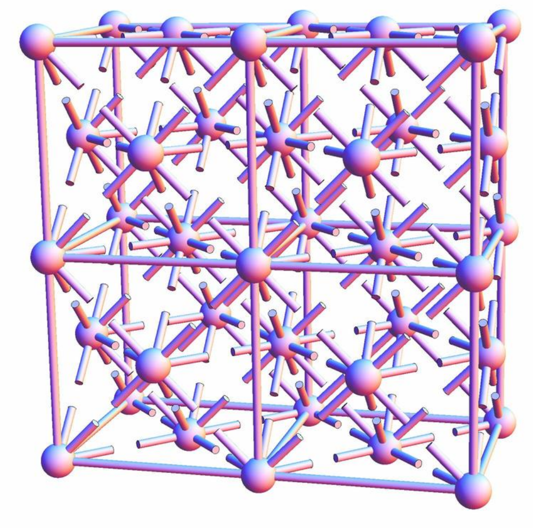

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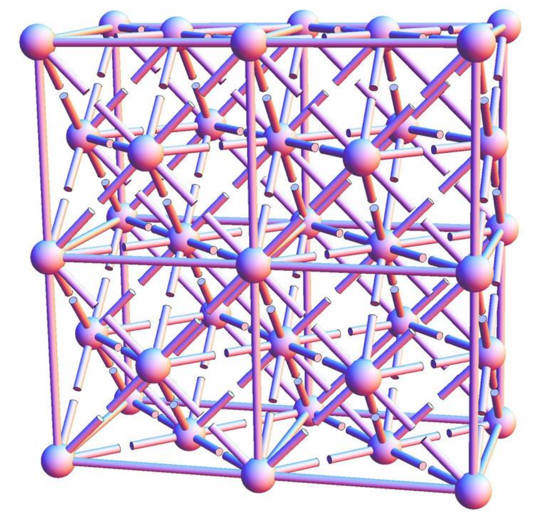

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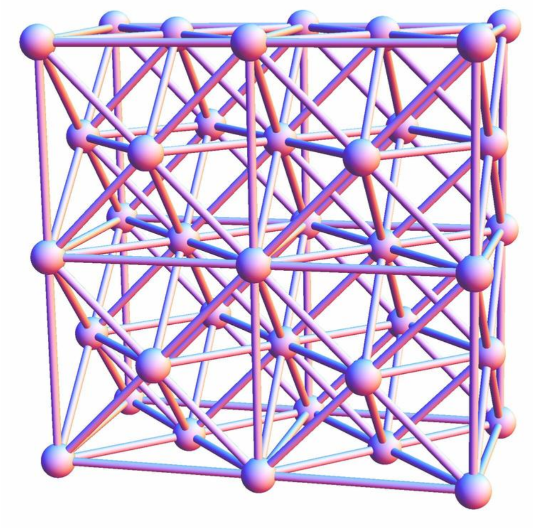

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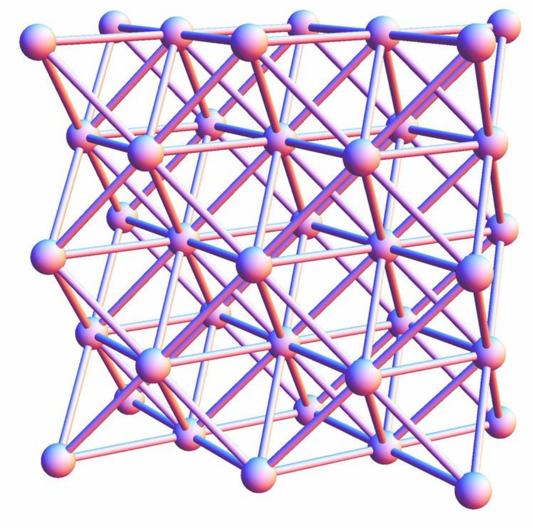

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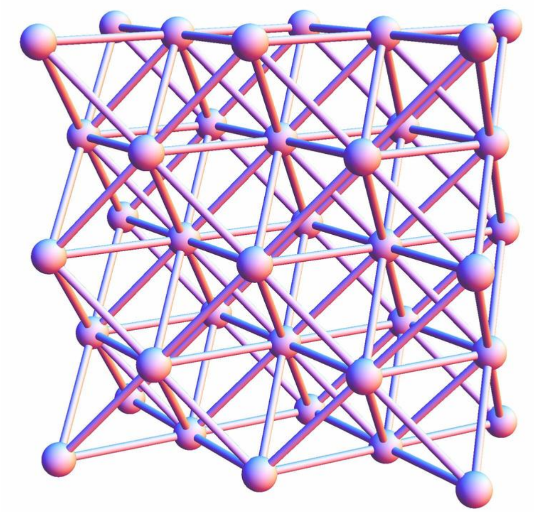

Generalization to higher dimensions is straight-forward.

Lattice Green's Function

The lattice Green's function is the probability generating function

$$P(\boldsymbol{x}; z) = \sum_{n=0}^{\infty} p_n(\boldsymbol{x}) z^n$$

where $p_n(\boldsymbol{x})$ is the probability of being at point $\boldsymbol{x}$ after n steps.

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Let $\lambda(\boldsymbol{k})$ denote the structure function of the lattice:

$$\lambda(\boldsymbol{k}) = \sum_{\boldsymbol{x} \in \mathbb{R}^d} p_1(\boldsymbol{x}) e^{i\boldsymbol{x}\cdot\boldsymbol{k}} = \binom{d}{2}^{-1} \sum_{1 \leqslant i < j \leqslant d} \cos(k_i)\cos(k_j).$$

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One is particularly interested in

$$P(\boldsymbol{0}; z) = \sum_{n=0}^{\infty} p_n(\boldsymbol{0}) z^n = \frac{1}{\pi^d} \int_0^{\pi} \cdots \int_0^{\pi} \frac{\mathrm{d}k_1 \ldots \mathrm{d}k_d}{1 - z\lambda(\boldsymbol{k})}$$

that encodes the return probability. It is a D-finite function, and its differential equation can be computed by creative telescoping.

Return Probability

Definition: The return probability R (Pólya number) is given by

$$R = 1 - \frac{1}{\sum_{n=0}^{\infty} p_n(\mathbf{0})} = 1 - \frac{1}{P(\mathbf{0};1)}.$$

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Fact: For $d = 3$, the return probability is one of Watson's integrals:

$$R_3 = 1 - \left(\frac{1}{\pi^3}\int_0^{\pi}\int_0^{\pi}\int_0^{\pi}\frac{\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3}{1-\frac{1}{3}(c_1c_2+c_1c_3+c_2c_3)}\right)^{-1} = 1 - \frac{16\sqrt[3]{4}\pi^4}{9(\Gamma(\frac{1}{3}))^6}$$

$$= 0.256318236504648771095030180 63\ldots \quad \text{where } c_i = \cos(k_i).$$

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$$= 0.2563182365046487710950301 8063\ldots \quad \text{where } c_i = \cos(k_i).$$

Results: for higher dimensions one approximates R using the ODE:

- $d = 4$: $R_4 = 0.09571315417256289673531676490 1210185\ldots$
- $d = 5$: $R_5 = 0.04657695746384802419337442059 4803291\ldots$
- $d = 6$: $R_6 = 0.02699987828795612426936417542 6196380\ldots$

Intermediate Conclusion

Significance of WZ theory and holonomic systems approach:

- Automatability
- Generality
- Shift from ad hoc to algorithmic
- Algorithm replaces ingenuity (or augments it)
- Can handle a quite large class of functions (= holonomic), even those that do not have a name.

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Drawbacks:

- Such proofs do not provide any “insight” (combinatorial interpretation, etc.).
- Not fully automated: certain technical details have to be checked manually (initial values, singularities, etc.).

Plane Partitions

Definition: A plane partition π of $n \in \mathbb{N}$ is

- a two-dimensional array $\pi = (\pi_{i,j})_{1 \leqslant i,j}$
- s.t. $\pi_{i,j} \in \mathbb{N}$ with finite sum $\sum \pi_{i,j} = n =: |\pi|$
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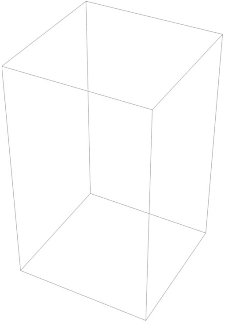

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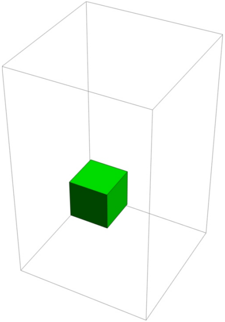

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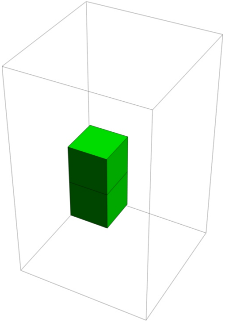

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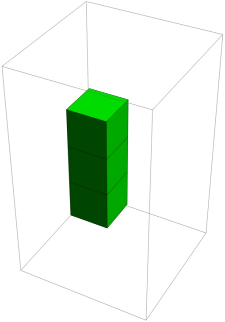

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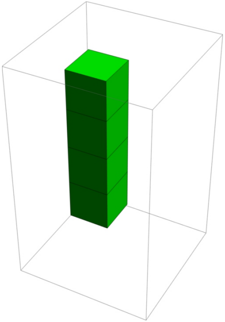

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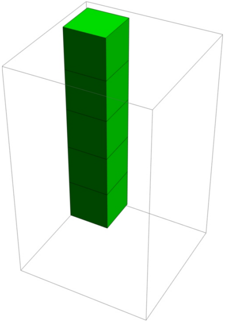

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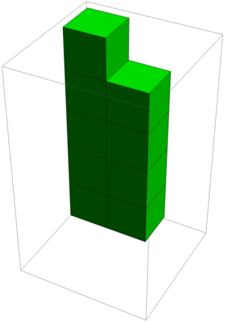

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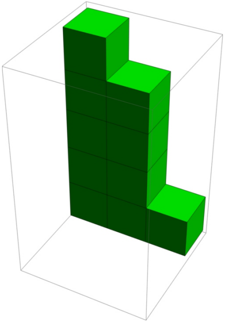

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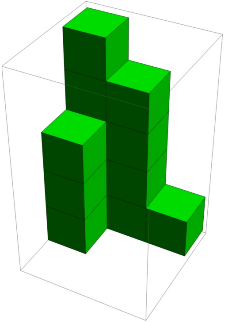

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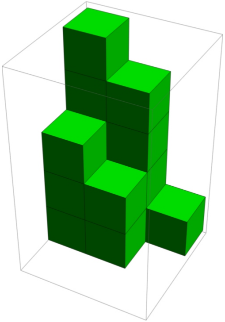

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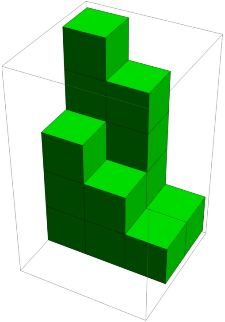
- and $\pi_{i,j} \geqslant \pi_{i+1,j}$ and $\pi_{i,j} \geqslant \pi_{i,j+1}$ for all $i, j \geqslant 1$.

Example: A plane partition π of 17 and its 3D Ferrers diagram:

5	4	1
3	2	1
1		



Plane Partitions

Definition: A plane partition π of $n \in \mathbb{N}$ is

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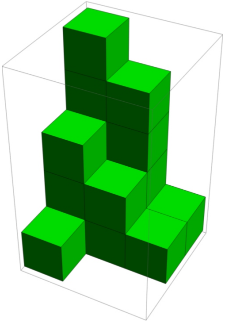
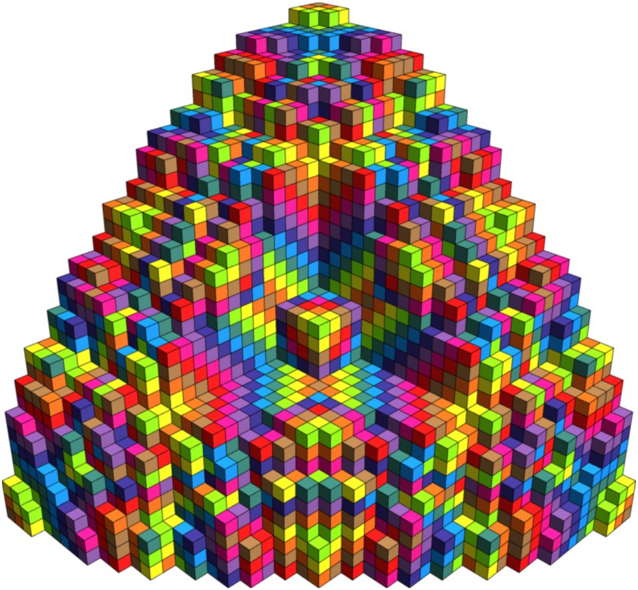


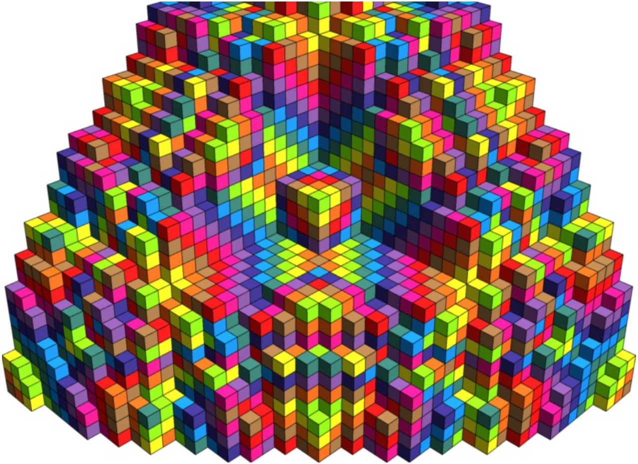
Totally Symmetric Plane Partitions



Totally Symmetric Plane Partitions

Conjecture 7. (see [11, Case 4]). The number of totally symmetric plane partitions with largest part $\leq$ n is equal to

$$T_n = \prod_{1\leq i\leq j\leq k\leq n} \frac{i+j+k-1}{i+j+k-2} .$$



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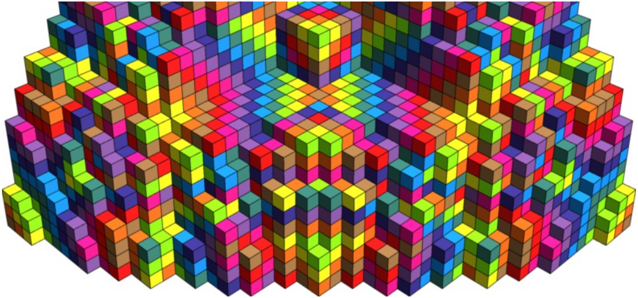
Note. All quantities arising in connection with Conjecture 7 have natural q-analogues. The q-analogue of T_n is

$$T_n(q) = \prod_{1 \le i \le j \le k \le n} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}$$



Orbit-Counting Generating Function for TSPPs

q-TSPP conjecture: $$\sum_{\pi \in \mathrm{TSPP}(n)} q^{|\pi/S_3|} = \prod_{1 \leqslant i \leqslant j \leqslant k \leqslant n} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}$$

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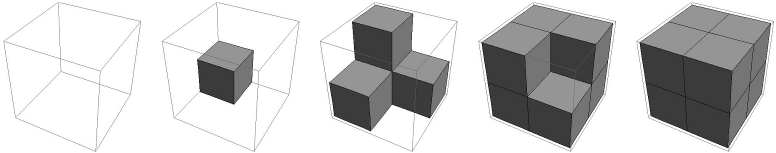
Example: For $n = 2$ there are five such TSPPs



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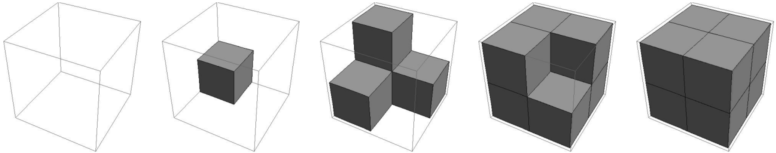

q^0

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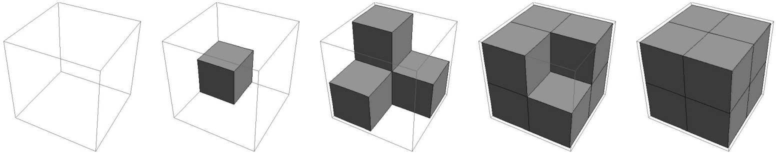

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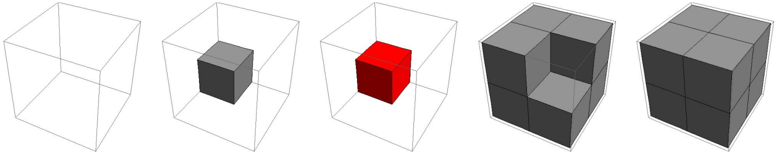

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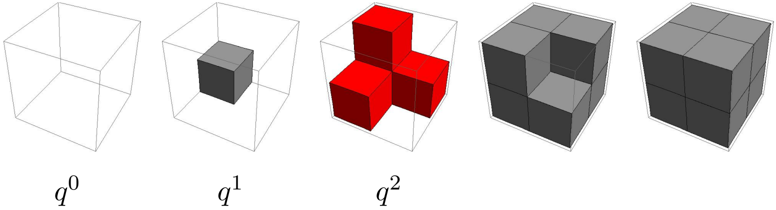

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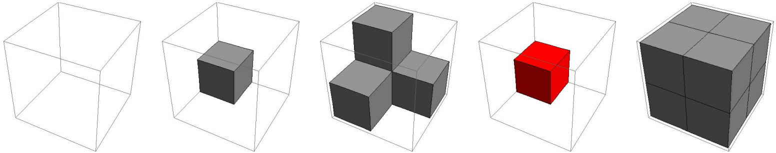

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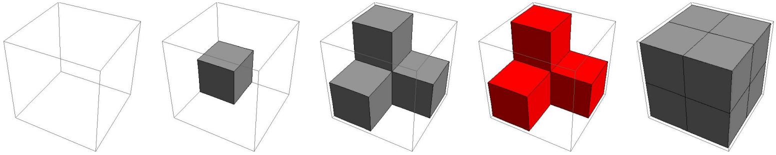

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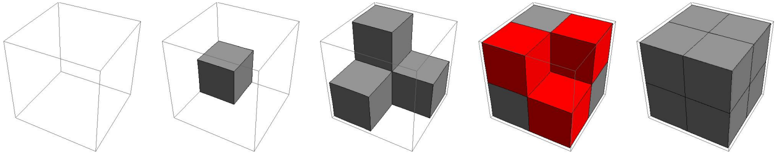

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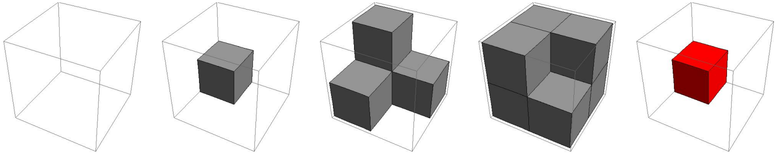

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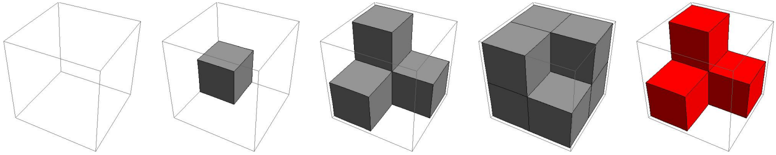

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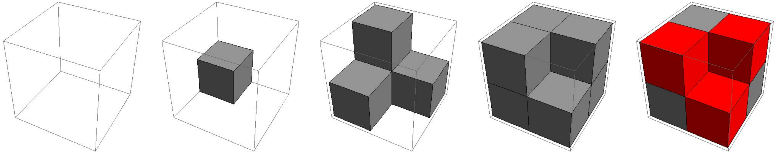

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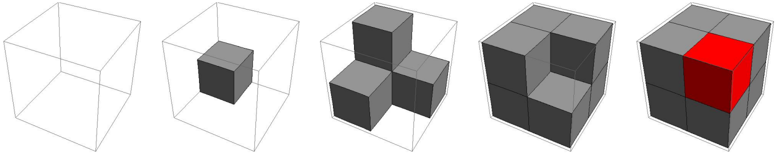

q^0 q^1 q^2 q^3 q^4

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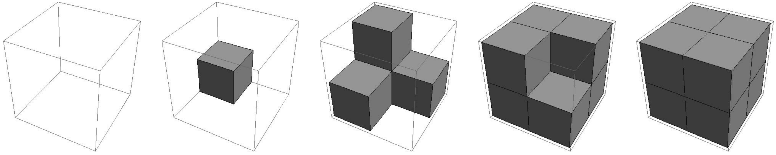

$$q^0 + q^1 + q^2 + q^3 + q^4$$
$$= \frac{1-q^5}{1-q}$$

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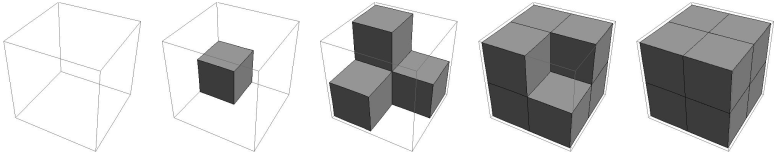


$$q^0 \quad + \quad q^1 \quad + \quad q^2 \quad + \quad q^3 \quad + \quad q^4$$

$$= \frac{1-q^5}{1-q} = \frac{1-q^2}{1-q} \cdot \frac{1-q^3}{1-q^2} \cdot \frac{1-q^4}{1-q^3} \cdot \frac{1-q^5}{1-q^4}$$

Determinantal Formulation

On the Generating Functions for Certain Classes of Plane Partitions

SOICHI OKADA

Department of Mathematics, University of Tokyo
Hongo, Tokyo, 113, Japan

Communicated by George Andrews

Received November 2, 1987

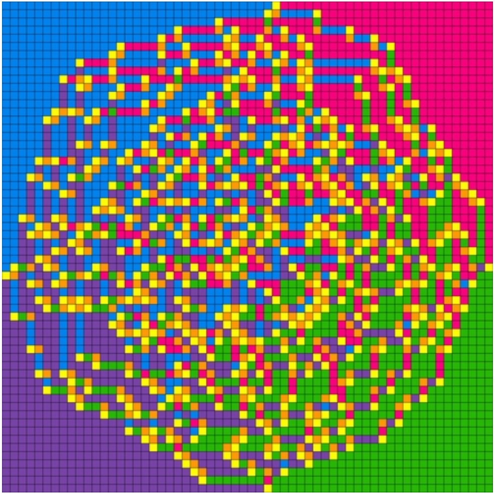
Okada's Theorem: The q-TSPP conjecture is true if

$$\det\left(a_{i,j}\right)_{1\leqslant i,j\leqslant n} = \prod_{1\leqslant i\leqslant j\leqslant k\leqslant n}\left(\frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}\right)^2, \qquad \text{where}$$

$$a_{i,j} := q^{i+j-1}\left(\begin{bmatrix} i+j-2 \\ i-1 \end{bmatrix}_q + q\begin{bmatrix} i+j-1 \\ i \end{bmatrix}_q\right) + (1+q^i)\delta_{i,j} - \delta_{i,j+1}.$$

Results on DSASMs and OSASMs

(joint work with Roger Behrend and Ilse Fischer)



Alternating Sign Matrices

Definition:

- quadratic matrix $(n \times n)$ with entries 0, 1, and -1
- 1's and -1's alternate along rows and along columns
- all row sums and all column sums equal 1

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DSASMs for $n = 1, 2, 3, 4$

$$\begin{pmatrix}1\end{pmatrix} \qquad \begin{pmatrix}1&0\\0&1\end{pmatrix} \quad \begin{pmatrix}0&1\\1&0\end{pmatrix}$$

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix} \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix} \begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix} \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$$

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix} \begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix} \begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&-1&1\\0&0&1&0\end{pmatrix}$$

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Number of ASMs

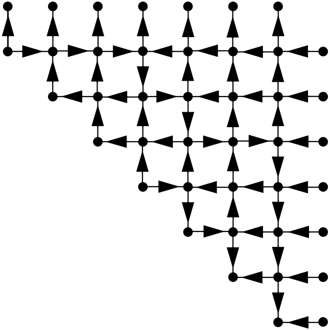
$$
\begin{aligned}
n = 1 &: 1 &&= 1 \\
n = 2 &: 2 &&= 2 \\
n = 3 &: 7 &&= 7 \\
n = 4 &: 42 &&= 2 \cdot 3 \cdot 7 \\
n = 5 &: 429 &&= 3 \cdot 11 \cdot 13 \\
n = 6 &: 7436 &&= 2^2 \cdot 11 \cdot 13^2 \\
n = 7 &: 218348 &&= 2^2 \cdot 13^2 \cdot 17 \cdot 19 \\
n = 8 &: 10850216 &&= 2^3 \cdot 13 \cdot 17^2 \cdot 19^2 \\
n = 9 &: 911835460 &&= 2^2 \cdot 5 \cdot 17^2 \cdot 19^3 \cdot 23 \\
n = 10 &: 129534272700 &&= 2^2 \cdot 3 \cdot 5^2 \cdot 7 \cdot 17 \cdot 19^3 \cdot 23^2 \\
n = 11 &: 31095744852375 &&= 3^2 \cdot 5^3 \cdot 7 \cdot 19^2 \cdot 23^3 \cdot 29 \cdot 31 \\
n = 12 &: 12611311859677500 &&= 2^2 \cdot 3^3 \cdot 5^4 \cdot 19 \cdot 23^3 \cdot 29^2 \cdot 31^2 \\
n = 13 &: 8639383518297652500 &&= 2^2 \cdot 3^5 \cdot 5^4 \cdot 23^2 \cdot 29^3 \cdot 31^3 \cdot 37 \\
n = 14 &: 9995541355448167482000 &&= 2^4 \cdot 3^5 \cdot 5^3 \cdot 23 \cdot 29^4 \cdot 31^4 \cdot 37^2 \\
n = 15 &: 19529076234661277104897200 &&= 2^4 \cdot 3^3 \cdot 5^2 \cdot 29^4 \cdot 31^5 \cdot 37^3 \cdot 41 \cdot 43
\end{aligned}
$$

Number of DSASMs

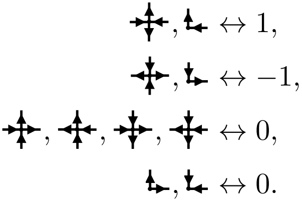
$$
\begin{aligned}
n=1&: 1 &&= 1\\
n=2&: 2 &&= 2\\
n=3&: 5 &&= 5\\
n=4&: 16 &&= 2^4\\
n=5&: 67 &&= 67\\
n=6&: 368 &&= 2^4\cdot 23\\
n=7&: 2630 &&= 2\cdot 5\cdot 263\\
n=8&: 24376 &&= 2^3\cdot 11\cdot 277\\
n=9&: 293770 &&= 2\cdot 5\cdot 29\cdot 1013\\
n=10&: 4610624 &&= 2^6\cdot 61\cdot 1181\\
n=11&: 94080653 &&= 4679\cdot 20107\\
n=12&: 2492747656 &&= 2^3\cdot 7\cdot 2063\cdot 21577\\
n=13&: 85827875506 &&= 2\cdot 29\cdot 73\cdot 20271109\\
n=14&: 3842929319936 &&= 2^{13}\cdot 7\cdot 67015369\\
n=15&: 223624506056156 &&= 2^2\cdot 67\cdot 7547\cdot 110563111\\
n=16&: 16901839470598576 &&= 2^4\cdot 13\cdot 12343\cdot 6583394929\\
n=17&: 1659776507866213636 &&= 2^2\cdot 263\cdot 1577734323066743\\
n=18&: 211853506422044996288 &&= 2^6\cdot 13\cdot 254631618295727159\\
n=19&: 35137231473111223912310 &&= 2\cdot 5\cdot 1601\cdot 2194705276271781631\\
n=20&: 7569998079873075147860464 &&= 2^4\cdot 473124879992067196741279
\end{aligned}
$$

Six-vertex model

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$



- The degree-4 vertices have two incoming and two outgoing edges.
- The top vertical edges point up.
- The rightmost horizontal edges point to the left.



Pfaffian formula for DSASMs

Theorem. The number of $(n \times n)$-DSASMs is equal to

$$\mathrm{Pf}_{\epsilon(n) \le i < j \le n-1}\left([u^i v^j] \frac{(v-u)(2+uv)}{(1-uv)(1-u-v)}\right),$$

where $\epsilon(n) = 0$ for even n and $\epsilon(n) = 1$ for odd n.

Off-Diagonally Symmetric Alternating Sign Matrices

Theorem (Kuperberg):

$$|\operatorname{OSASM}(2n)| = \prod_{i=1}^{n} \frac{(6i-2)!}{(2n+2i)!}.$$

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Theorem: The number of off-diagonally symmetric alternating sign matrices, $|\operatorname{OSASM}(n)|$, is given by

$$\operatorname{Pf}_{0\leqslant i<j\leqslant n-\chi_{\text{even}}(n)} \left(\begin{cases} [u^i v^j] \dfrac{v-u}{(1-uv)(1-u-v)}, & j \leqslant n-1 \\ (-1)^i, & j=n \end{cases} \right).$$

Request by Zeilberger (dated June 23, 2021)

Von	Doron Zeilberger
An	Christoph Koutschan (RICAM)
Kopie (CC)	Di Francesco, Philippe
Betreff	**challenge**

Dear Christoph,

Philippe Di Francesco just gave a great talk at the Lattice path conference mentioning, inter alia, a certain conjectured determinant.

It is

Conj. 8.1 (combined with Th. 8.2) in
https://arxiv.org/pdf/2102.02920.pdf

I am curious if you can prove it by the Koutschan-Zeilberger-Aek holonomic ansatz method.

If you can do it before Friday, June 25, 2021, 17:00 Paris time, I will mention it in my talk in that conference.

Best wishes

Doron

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Conjecture (Di Francesco's determinant for 20V configurations):

$$\det_{0\leqslant i,j<n}\left(2^i\binom{i+2j+1}{2j+1}-\binom{i-1}{2j+1}\right)=2\prod_{i=1}^{n}\frac{2^{i-1}\,(4i-2)!}{(n+2i-1)!}$$

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Determinants and Pfaffians

Who you gonna call?

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Who you gonna call?

ADVANCED DETERMINANT CALCULUS

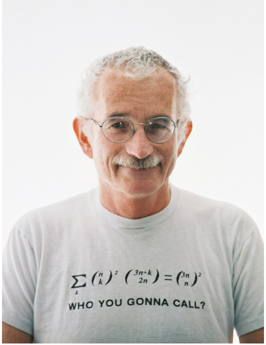
C. KRATTENTHALER†

Institut für Mathematik der Universität Wien,
Strudlhofgasse 4, A-1090 Wien, Austria.
E-mail: `kratt@pap.univie.ac.at`
WWW: `http://radon.mat.univie.ac.at/People/kratt`

Dedicated to the pioneer of determinant evaluations (among many other things), George Andrews

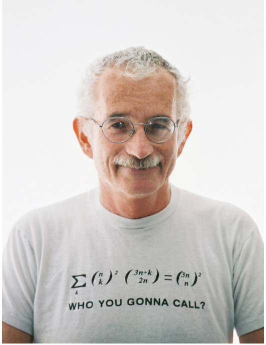
The Holonomic Ansatz

The Holonomic Ansatz



The Holonomic Ansatz

Problem: Prove a determinantal identity of the form $\det\limits_{1\leqslant i,j\leqslant n}(a_{i,j}) = b_n$



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Problem: Prove a determinantal identity of the form $\det\limits_{1\leqslant i,j\leqslant n}(a_{i,j}) = b_n$, where

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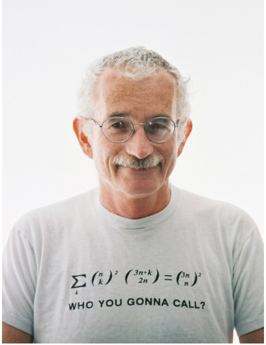

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Problem: Prove a determinantal identity of the form $\det_{1 \leqslant i,j \leqslant n}(a_{i,j}) = b_n$, where

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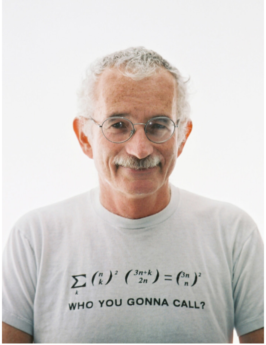
- that does not depend on n



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Problem: Prove a determinantal identity of the form $\det_{1\leqslant i,j\leqslant n}(a_{i,j}) = b_n$, where

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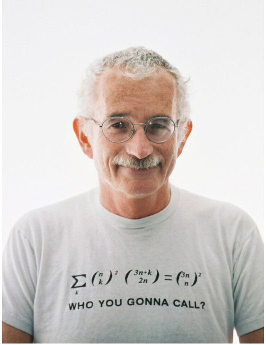
- b_n is a closed form ($b_n \neq 0$ for all n).



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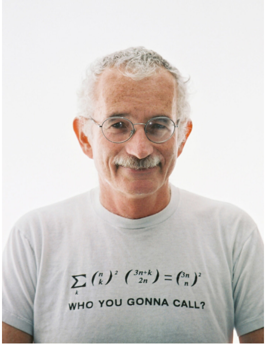

$$\mathcal{A}_n = \left(\begin{array}{ccc|c} & \mathcal{A}_{n-1} & & \\ \hline a_{n,1} & \cdots & a_{n,n-1} & a_{n,n} \end{array}\right)$$

Laplace expansion:

$$\det(\mathcal{A}_n) = a_{n,1}\mathrm{Cof}_{n,1} + \cdots + a_{n,n-1}\mathrm{Cof}_{n,n-1} + a_{n,n}\det(\mathcal{A}_{n-1})$$

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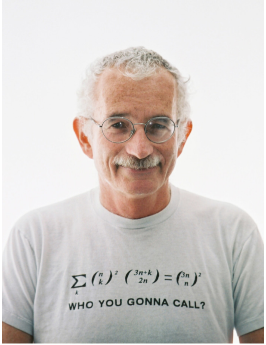

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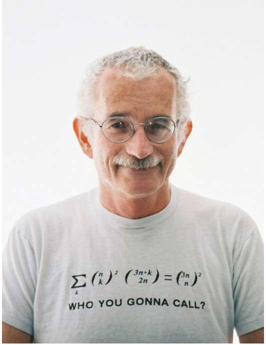

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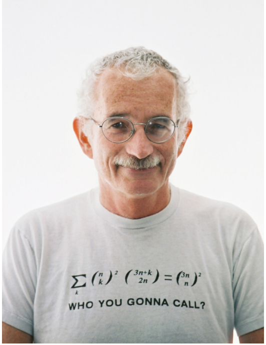

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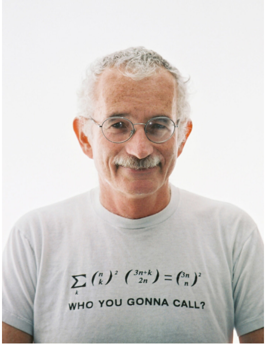

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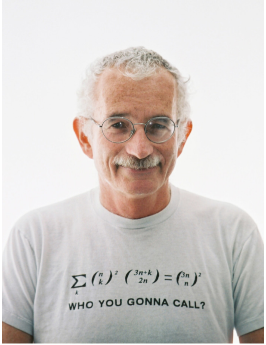

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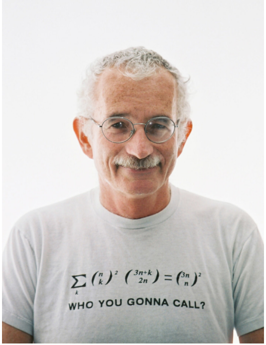

Laplace expansion:

$$0 = \sum_{j=1}^{n} a_{i,j} c_{n,j} \quad (1 \leqslant i < n), \quad c_{n,n} = 1$$

Recipe for the Holonomic Ansatz

1. Guess a set of recurrences (holonomic description) for the normalized cofactors $c_{n,j}$.

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$$c_{n,n} = 1 \qquad (1 \leqslant n) \tag{1}$$

$$\sum_{j=1}^{n} a_{i,j} c_{n,j} = 0 \qquad (1 \leqslant i < n) \tag{2}$$

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Justification: Identity Found by Proving Identities!

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Di Francesco's Determinant

Conjecture (Di Francesco's determinant for 20V configurations):

$$\det_{0 \leqslant i,j < n} \left(2^i \binom{i+2j+1}{2j+1} - \binom{i-1}{2j+1} \right) = 2 \prod_{i=1}^{n} \frac{2^{i-1}\,(4i-2)!}{(n+2i-1)!}$$

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Proof:

- Compute data for $c_{n,j}$ for $0\leqslant j<n\leqslant 30$

1										
-1	1									
1	-2	1								
$-\frac{16}{15}$	$\frac{47}{15}$	$-\frac{46}{15}$	1							
$\frac{16}{13}$	$-\frac{60}{13}$	$\frac{85}{13}$	$-\frac{54}{13}$	1						
$-\frac{20}{13}$	$\frac{88}{13}$	$-\frac{633}{52}$	$\frac{291}{26}$	$-\frac{21}{4}$	1					
$\frac{2008}{969}$	$-\frac{9808}{969}$	$\frac{2441}{114}$	$-\frac{8107}{323}$	$\frac{33115}{1938}$	$-\frac{362}{57}$	1				
$-\frac{10592}{3553}$	$\frac{55360}{3553}$	$-\frac{7712}{209}$	$\frac{16567}{323}$	$-\frac{159022}{3553}$	$\frac{5062}{209}$	$-\frac{82}{11}$	1			
$\frac{2608}{575}$	$-\frac{2848}{115}$	$\frac{36496}{575}$	$-\frac{57388}{575}$	$\frac{59828}{575}$	$-\frac{41696}{575}$	$\frac{18739}{575}$	$-\frac{214}{25}$	1		
$-\frac{32432}{4485}$	$\frac{182176}{4485}$	$-\frac{12656}{115}$	$\frac{849728}{4485}$	$-\frac{1011076}{4485}$	$\frac{56467}{299}$	$-\frac{492191}{4485}$	$\frac{8228}{195}$	$-\frac{29}{3}$	1	
$\frac{161632}{13485}$	$-\frac{924992}{13485}$	$\frac{2606624}{13485}$	$-\frac{4799104}{13485}$	$\frac{1262497}{2697}$	$-\frac{6078586}{13485}$	$\frac{4266601}{13485}$	$-\frac{425608}{2697}$	$\frac{47679}{899}$	$-\frac{334}{31}$	1

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Proof:

- Compute data for $c_{n,j}$ for $0\leqslant j<n\leqslant 30$
- Guess recurrences for $c_{n,j}$ (using Manuel Kauers' `Guess.m`):

$$\begin{aligned}
&\{-(1+j)(6+2j-n)(7+2j-n)(3+j+n)(1+3n)\left(24j+88j^2+126j^3+88j^4+30j^5+4j^6-27n-108jn-168j^2n-132j^3n-52j^4n-8j^5n+18n^2+60jn^2+72j^2n^2+40j^3n^2+8j^4n^2\right)S_j^2+\\
&3(2+j)\left(21+56j+46j^2+16j^3+2j^4\right)(j-n)n^2(1+2n)(-1+3n)(1+3n)S_n+\\
&2(1+j)(1+3n)\left(1872j+9168j^2+19060j^3+21952j^4+15320j^5+6640j^6+1748j^7+256j^8+16j^9-2106n-11580jn-27220j^2n-35864j^3n-28700j^4n-14184j^5n-4220j^6n-692j^7n-48j^8n+1986n^2+9197jn^2+17672j^2n^2+18604j^3n^2+\right.\\
&\left.11396j^4n^2+4032j^5n^2+764j^6n^2+60j^7n^2+111n^3+469jn^3+820j^2n^3+768j^3n^3+408j^4n^3+112j^5n^3+12j^6n^3-123n^4-458jn^4-672j^2n^4-496j^3n^4-180j^4n^4-24j^5n^4+78n^5+248jn^5+280j^2n^5+144j^3n^5+24j^4n^5\right)S_j-\\
&(1+2j+n)\left(720j+3648j^2+7844j^3+9340j^4+6736j^5+3016j^6+820j^7+124j^8+8j^9-810n-3414jn-4346j^2n+1316j^3n+9104j^4n+10452j^5n+6048j^6n+1954j^7n+336j^8n+24j^9n-645n^2-\right.\\
&7592jn^2-26703j^2n^2-45406j^3n^2-43318j^4n^2-24394j^5n^2-8054j^6n^2-1442j^7n^2-108j^8n^2+4146n^3+22215jn^3+48959j^2n^3+56710j^3n^3+37202j^4n^3+13922j^5n^3+\\
&\left.2774j^6n^3+228j^7n^3-2721n^4-12651jn^4-22382j^2n^4-19296j^3n^4-8800j^4n^4-2052j^5n^4-192j^6n^4+1542n^5+4898jn^5+5588j^2n^5+3072j^3n^5+832j^4n^5+88j^5n^5\right),\\
&-3(1+j)(3+j+n)(-1+3n)(1+3n)\left(24j+88j^2+126j^3+88j^4+30j^5+4j^6-27n-108jn-168j^2n-132j^3n-52j^4n-8j^5n+18n^2+60jn^2+72j^2n^2+40j^3n^2+8j^4n^2\right)S_nS_j+\\
&3(2+j)(j-n)(-1+3n)(1+3n)\left(72+288j+466j^2+390j^3+178j^4+42j^5+4j^6+81n+288jn+372j^2n+228j^3n+68j^4n+8j^5n+54n^2+156jn^2+144j^2n^2+56j^3n^2+8j^4n^2\right)S_n+\\
&2(1+j)(5+2j-n)(-1+4n)(1+4n)\left(36+186j+380j^2+404j^3+232j^4+68j^5+8j^6+45n+168jn+240j^2n+172j^3n+60j^4n+8j^5n+9n^2+30jn^2+36j^2n^2+20j^3n^2+4j^4n^2\right)S_j-\\
&2(2+j)(1+2j+n)(-1+4n)(1+4n)\left(90+366j+632j^2+580j^3+292j^4+76j^5+8j^6-27n-132jn-228j^2n-172j^3n-60j^4n-8j^5n+27n^2+78jn^2+72j^2n^2+28j^3n^2+4j^4n^2\right),\\
&9(-1+j-n)(1+n)(3+j+n)(3+2n)(-1+3n)(1+3n)^2(2+3n)(4+3n)\left(24j+88j^2+126j^3+88j^4+30j^5+4j^6-27n-108jn-168j^2n-132j^3n-52j^4n-8j^5n+18n^2+60jn^2+72j^2n^2+40j^3n^2+8j^4n^2\right)S_n^2-\\
&6(-1+3n)(1+3n)\left(-7776j-37152j^2-67608j^3-46704j^4+20400j^5+62688j^6+50568j^7+20784j^8+4416j^9+384j^{10}+7452n-11664jn-171684j^2n-419220j^3n-423960j^4n-95880j^5n+199776j^6n+217644j^7n+99624j^8n+22416j^9n+2016j^{10}n+\right.\\
&45846n^2+137178jn^2-49626j^2n^2-698252j^3n^2-1160368j^4n^2-829556j^5n^2-162976j^6n^2+150088j^7n^2+113888j^8n^2+30744j^9n^2+3024j^{10}n^2+92826n^3+444129jn^3+709672j^2n^3+196482j^3n^3-783488j^4n^3-1136566j^5n^3-\\
&696552j^6n^3-193240j^7n^3-4800j^8n^3+8856j^9n^3+1296j^{10}n^3+67662n^4+448944jn^4+1099060j^2n^4+1296068j^3n^4+682760j^4n^4-45596j^5n^4-265128j^6n^4-145072j^7n^4-33104j^8n^4-2592j^9n^4-2142n^5+92601jn^5+425078j^2n^5+\\
&781090j^3n^5+755604j^4n^5+411194j^5n^5+117540j^6n^5+11968j^7n^5-672j^8n^5-10836n^6-61056jn^6-108784j^2n^6-51056j^3n^6+46096j^4n^6+67744j^5n^6+29248j^6n^6+4224j^7n^6+18504n^7+34464jn^7-20256j^2n^7-79264j^3n^7-\\
&\left.66208j^4n^7-19840j^5n^7-1536j^6n^7+18288n^8+56640jn^8+58752j^2n^8+23360j^3n^8-1472j^4n^8-1920j^5n^8+4320n^9+14400jn^9+17280j^2n^9+9600j^3n^9+1920j^4n^9\right)S_n-12(1+j)(4+2j-n)(5+2j-n)(1+3n)(4+3n)(-1+4n)(1+4n)\\
&\left(-54-288j-576j^2-464j^3+100j^4+488j^5+380j^6+128j^7+16j^8-135n-648jn-1284j^2n-1408j^3n-912j^4n-336j^5n-56j^6n-108n^2-456jn^2-788j^2n^2-752j^3n^2-428j^4n^2-144j^5n^2-24j^6n^2-27n^3-96jn^3-128j^2n^3-80j^3n^3-20j^4n^3\right)S_j+\\
&4(1+2j+n)(4+3n)(-1+4n)(1+4n)\left(-3240-16956j-37044j^2-39072j^3-10224j^4+23592j^5+32280j^6+19848j^7+6792j^8+1248j^9+96j^{10}-11988n-63612jn-148950j^2n-191700j^3n-129264j^4n-\right.\\
&15168j^5n+48816j^6n+43164j^7n+17304j^8n+3504j^9n+288j^{10}n-6642n^2-34137jn^2-93402j^2n^2-180616j^3n^2-246480j^4n^2-225316j^5n^2-132048j^6n^2-46924j^7n^2-9072j^8n^2-720j^9n^2+\\
&162n^3+8463jn^3+48482j^2n^3+112656j^3n^3+137240j^4n^3+94480j^5n^3+36088j^6n^3+6768j^7n^3+432j^8n^3-1350n^4-12231jn^4-34770j^2n^4-45488j^3n^4-29376j^4n^4-7544j^5n^4+696j^6n^4+\\
&\left.472j^7n^4+1170n^5+4113jn^5+2932j^2n^5-4516j^3n^5-8416j^4n^5-5068j^5n^5-1024j^6n^5+1728jn^6+5760j^2n^6+6912j^3n^6+3840j^4n^6+768j^5n^6-576n^7-1920jn^7-2304j^2n^7-1280j^3n^7-256j^4n^7\right)\}
\end{aligned}$$

Di Francesco's Determinant

Conjecture (Di Francesco's determinant for 20V configurations):

$$\det_{0\leqslant i,j<n}\left(2^i\binom{i+2j+1}{2j+1}-\binom{i-1}{2j+1}\right)=2\prod_{i=1}^{n}\frac{2^{i-1}\,(4i-2)!}{(n+2i-1)!}$$

Proof:

- Compute data for $c_{n,j}$ for $0\leqslant j<n\leqslant 30$
- Guess recurrences for $c_{n,j}$ (using Manuel Kauers' `Guess.m`):
- Derive diagonal recurrence for $c_{n,n}$; it implies $c_{n,n}=1$.

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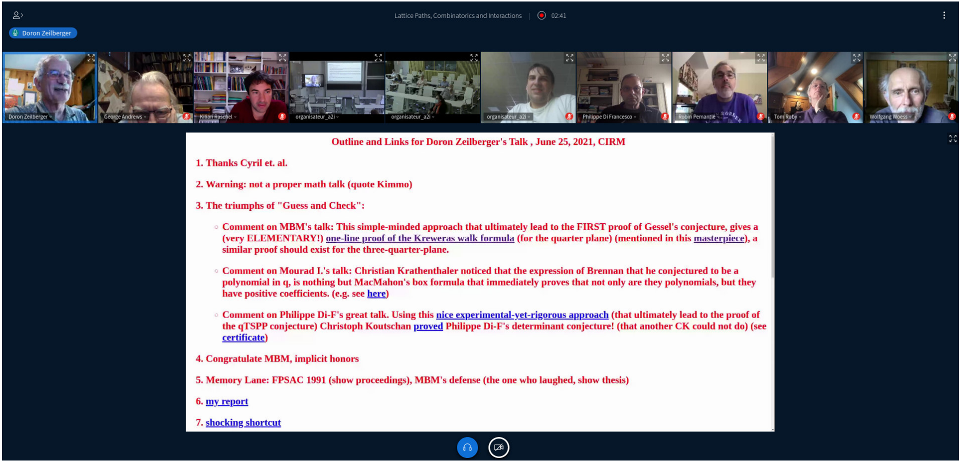
Zeilberger's Talk at the Lattice Path Conference



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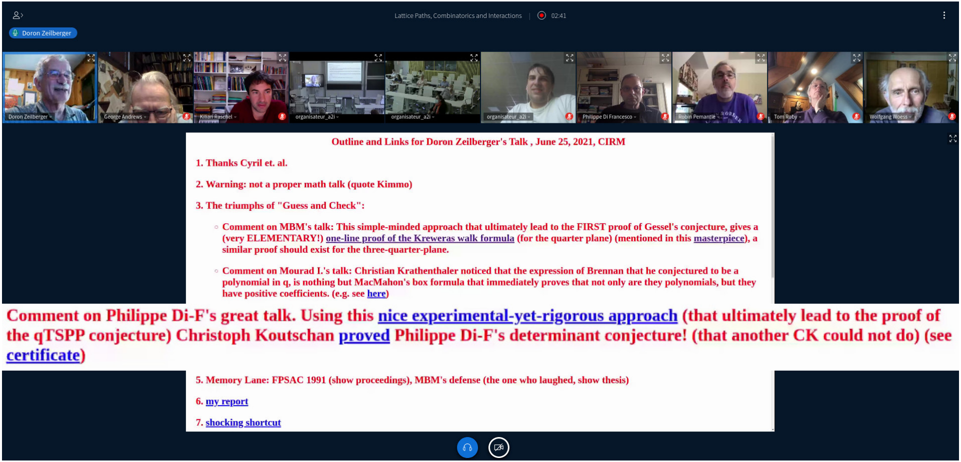

Ternary Variations of Di Francesco's Determinant

(joint work with Christian Krattenthaler and Michael Schlosser)

$$\det_{0\le i,j\le n-1}\left(3^i\binom{i+3j}{3j}+\binom{-i+3j}{3j}\right)$$

$$=\begin{cases} 2^{9\binom{m}{2}+3m+1}3^{9\binom{m}{2}+3m}\dfrac{(\frac{1}{6})_m}{(\frac{7}{12})_m}\displaystyle\prod_{i=1}^{3m-1}\frac{(4i)!}{(3i)!}\prod_{i=1}^{m}\frac{(3i-2)!}{(12i-8)!}, & \textit{for } n=3m,\\ 2^{9\binom{m}{2}+6m+1}3^{9\binom{m}{2}+6m}\dfrac{(\frac{1}{2})_m}{(\frac{11}{12})_m}\displaystyle\prod_{i=1}^{3m}\frac{(4i)!}{(3i)!}\prod_{i=1}^{m}\frac{(3i-1)!}{(12i-4)!}, & \textit{for } n=3m+1,\\ 2^{9\binom{m}{2}+9m+2}3^{9\binom{m}{2}+9m}\dfrac{(\frac{5}{6})_m}{(\frac{15}{12})_{m+1}}\displaystyle\prod_{i=1}^{3m+1}\frac{(4i)!}{(3i)!}\prod_{i=1}^{m}\frac{(3i)!}{(12i)!}, & \textit{for } n=3m+2,\end{cases}$$

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$$\det_{0\le i,j\le n-1}\left(3^i\binom{i+3j+1}{3j+1}+\binom{-i+3j+1}{3j+1}\right)$$
$$=\begin{cases} 2^{9\binom{m}{2}-6m+1}3^{9\binom{m}{2}+6m}\dfrac{\left(\frac{2}{3}\right)_{3m}}{\left(\frac{11}{12}\right)_m\left(\frac{1}{2}\right)_{2m}}\displaystyle\prod_{i=1}^{3m}\frac{(4i)!}{(3i)!}\prod_{i=1}^{m}\frac{(3i-1)!}{(12i-4)!}, & \textit{for } n=3m,\\ 2^{9\binom{m}{2}-3m-1}3^{9\binom{m}{2}+9m}\dfrac{\left(\frac{5}{3}\right)_{3m}}{\left(\frac{15}{12}\right)_m\left(\frac{7}{6}\right)_{2m}}\displaystyle\prod_{i=1}^{3m+1}\frac{(4i)!}{(3i)!}\prod_{i=1}^{m}\frac{(3i)!}{(12i)!}, & \textit{for } n=3m+1,\\ 2^{9\binom{m}{2}-6}3^{9\binom{m}{2}+12m+1}\dfrac{\left(\frac{8}{3}\right)_{3m}}{\left(\frac{7}{12}\right)_{m+1}\left(\frac{11}{6}\right)_{2m}}\displaystyle\prod_{i=1}^{3m+2}\frac{(4i)!}{(3i)!}\prod_{i=1}^{m}\frac{(3i+1)!}{(12i+4)!}, & \textit{for } n=3m+2,\end{cases}$$

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$$\det_{0\le i,j\le n-1}\left(3^i\binom{i+3j+2}{3j+2}+\binom{-i+3j+2}{3j+2}\right)$$

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Ternary Variations of Di Francesco's Determinant

Theorem: For $n \geqslant 1$ we have

$$\det_{0 \leqslant i,j < n}\left(3^i \binom{i+3j}{3j} + \binom{-i+3j}{3j}\right) = 2 \prod_{i=1}^{n} \frac{2^{i-1}\,\Gamma(4i-3)\,\Gamma\left(\frac{i+1}{3}\right)}{\Gamma(3i-2)\,\Gamma\left(\frac{4i-2}{3}\right)}$$

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Three Infinite Families

Conjecture: for all $x \in \mathbb{N}_0$ and for all $n \in \mathbb{N}, n \geqslant x$, we have

$$\det_{0 \leqslant i,j < n} \left(3^{i+x} \binom{i+3j-x}{3j} + \binom{-i+3j-3x}{3j} \right) =$$

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where

$$\Xi(x) := \prod_{i=2}^{x} \frac{3\,\Gamma(i)\,\Gamma(4i-3)\,\Gamma(4i-2)}{2\,\Gamma(3i-2)^2\,\Gamma(3i-1)}$$

$$\mu_m(x) := \begin{cases} 2, & \text{if } 3 \mid (x-m) \\ 1, & \text{otherwise.} \end{cases}$$

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$$\frac{\mu_0(x)}{n}\,\Xi(x)\,(-1)^{\lfloor\frac{x+1}{3}\rfloor}\prod_{i=2}^{n}\frac{2^{i-3}\,\Gamma(4i+1)\,\Gamma\left(\frac{i-1}{3}\right)}{9\,\Gamma(3i)\,\Gamma\left(\frac{4i+2}{3}\right)}$$

Recent Progress on Some Conjectures

DOMINO TILINGS, NONINTERSECTING LATTICE PATHS AND SUBCLASSES OF KOUTSCHAN–KRATTENTHALER–SCHLOSSER DETERMINANTS

QIPIN CHEN, SHANE CHERN, AND ATSURO YOSHIDA

ABSTRACT. Koutschan, Krattenthaler and Schlosser recently considered a family of binomial determinants. In this work, we give combinatorial interpretations of two subclasses of these determinants in terms of domino tilings and nonintersecting lattice paths, thereby partially answering a question of theirs. Furthermore, the determinant evaluations established by Koutschan, Krattenthaler and Schlosser produce many product formulas for our weighted enumerations of domino tilings and nonintersecting lattice paths. However, there are still two enumerations left corresponding to conjectural formulas made by the three. We hereby prove the two conjectures using the principle of holonomic Ansatz plus the approach of modular reduction for creative telescoping, and hence fill the gap.

Families of Binomial Determinants

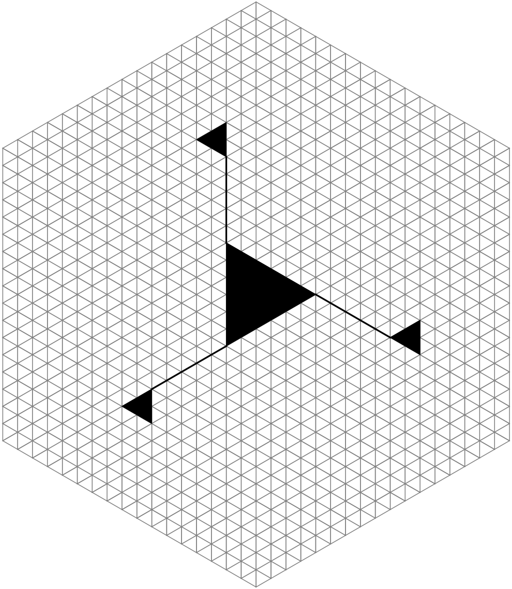
(joint work with Hao Du, Thotsaporn Thanatipanonda, and Elaine Wong)

Inspired by some conjectures in Christian Krattenthaler's "Advanced Determinant Calculus: A Complement".

Theorem: Let μ be an indeterminate and let $m, r \in \mathbb{Z}$.
If $m \geqslant r \geqslant 1$, then

$$\det_{1 \leqslant i,j \leqslant 2m+1} \left[\binom{\mu + i + j + 2r}{j + 2r - 2} - \delta_{i,j+2r} \right]$$
$$= \frac{(-1)^{m-r+1} (\mu+3) (m+r+1)_{m-r}}{2^{2m-2r+1} \left(\frac{\mu}{2} + r + \frac{3}{2}\right)_{m-r+1}} \cdot \prod_{i=1}^{2m} \frac{(\mu+i+3)_{2r}}{(i)_{2r}}$$
$$\times \prod_{i=1}^{m-r} \frac{(\mu+2i+6r+3)_i^2 \left(\frac{\mu}{2}+2i+3r+2\right)_{i-1}^2}{(i)_i^2 \left(\frac{\mu}{2}+i+3r+2\right)_{i-1}^2}.$$

Combinatorial Interpretation: Holey Hexagon



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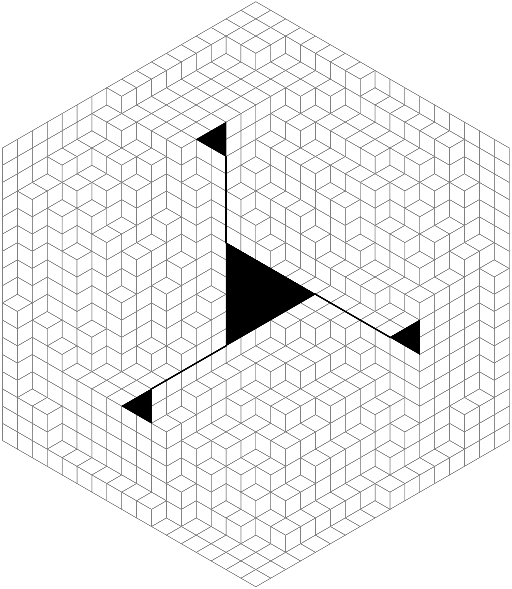

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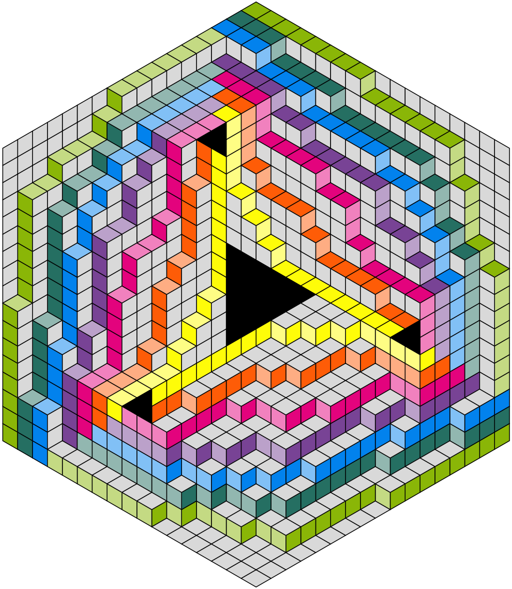

The q-Case

Consider q-difference equations involving the q-shift operation

$$x \mapsto qx, \quad \text{resp. } q^n \mapsto q^{n+1}.$$

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- q-special functions: q-Bessel functions, q-Legendre polynomials, q-Gegenbauer polynomials, etc.

q-TSPP: Holonomic Description of the Cofactors

The recurrences have the form

$$\bigcirc \cdot c_{n,j+4} = \bigcirc \cdot c_{n,j} + \bigcirc \cdot c_{n,j+1} + \bigcirc \cdot c_{n,j+2} + \bigcirc \cdot c_{n,j+3} + \bigcirc \cdot c_{n+2,j} + \bigcirc \cdot c_{n+2,j+1}$$

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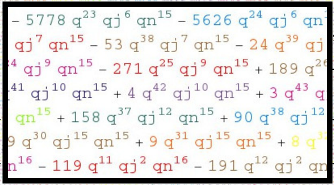

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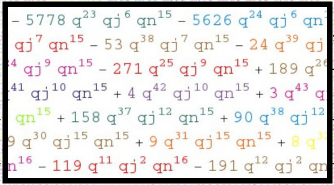

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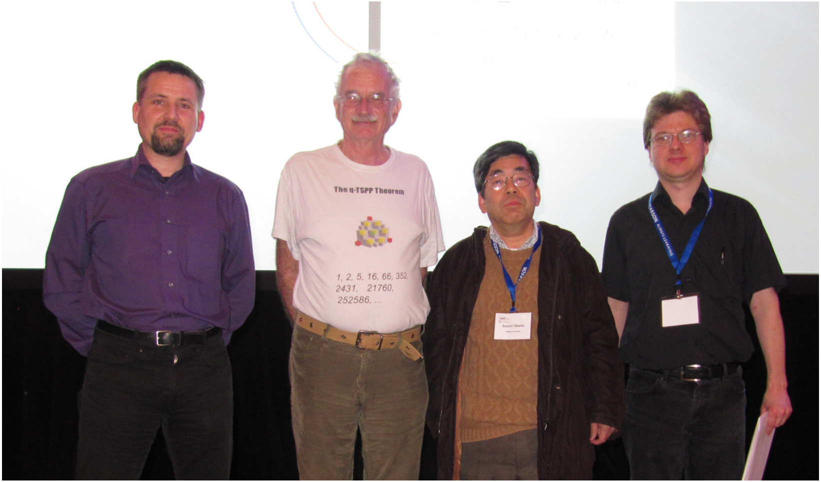
The total size is 244MB (several 1000 pages of paper)!

Solution of the q-TSPP Conjecture

(joint work with Manuel Kauers and Doron Zeilberger)



David P. Robbins Prize at the AMS Joint Meeting (Seattle, 2016)

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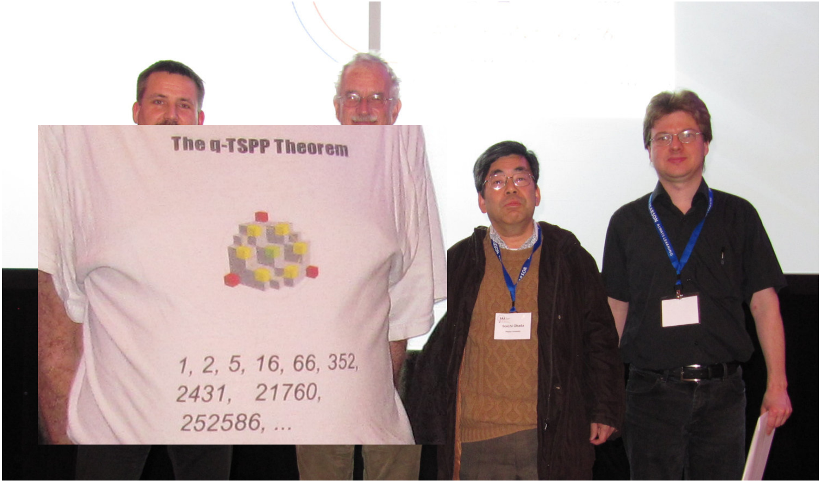

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Identity Found by Proving Identities

Identity proving is now a whole branch of symbolic computation:

- Binomial sums and other combinatorial identities, e.g.,

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{k+n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{k+n}{k} \sum_{j=0}^{k} \binom{k}{j}^3$$

- Special function identities (integrals or sums), e.g.,

$$\int_{-1}^{1} \left(1-x^2\right)^{\nu-\frac{1}{2}} e^{iax} C_n^{(\nu)}(x)\,\mathrm{d}x = \frac{\pi i^n \Gamma(n+2\nu) J_{n+\nu}(a)}{2^{\nu-1} a^\nu n!\,\Gamma(\nu)}$$

- Evaluations of symbolic determinants, e.g.,

$$\det_{0\leqslant i,j<n} \left(2^i \binom{i+2j+1}{2j+1} - \binom{i-1}{2j+1} \right) = 2 \prod_{i=1}^{n} \frac{2^{i-1}\,(4i-2)!}{(n+2i-1)!}$$