Unimodal Polynomials and Lattice Walk Enumeration with Experimental Mathematics

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Motivation

Humans have been counting for millennia:

- How many berry bushes remain to gather from,
- The number of people in their village,
- How long it has been since the water froze.

A single person could simply count the berries or people or days. The important jump for society was to communicate that number: tally marks.

Motivation

- The abacus arose to do simple calculations roughly 5,000 years ago.
- Gauss found a quick way to sum $1, \ldots, 100$ in the 1700s.
- Dr. Z. used recurrences inspired by O'Hara to show the q-binomials are unimodal[Zei89].
- Ayyer and Dr. Z. also used generating function relations to prove a result about polymers bounded between plates[AZ07].

Beyond the importance of answering the initial question, the examples all provide methods to solve more than just one problem.

Abstract

- Utilizing experimental mathematics to further our knowledge.
- With the aid of computers, we can create many families of unimodal polynomials automatically.
- Again through the speed of computers, we can succinctly describe lattice walks in very little time.

Overview

Topic Introduction

"The study of unimodality and log-concavity arise often in combinatorics, economics of uncertainty and information, and algebra, and have been the subject of considerable research." Alvarez et. al.[AAR00]

Definitions

Definition 1 (Unimodal)

A sequence $\mathcal{A} = \{a_0, \ldots, a_n\}$ is *unimodal* if it is weakly-increasing up to a point and then it is decreasing. I.e. there exists an index i such that $a_0 \leq a_1 \leq \cdots \leq a_i \geq \cdots \geq a_n$.

Definition 2 (Symmetric)

A sequence $\mathcal{A} = \{a_0, \ldots, a_n\}$ is *symmetric* if $a_i = a_{n-i}$ for every $0 \leq i \leq n$.

A polynomial is said to have the above properties if its sequence of coefficients does.

$$1 + 2q + 3q^2 + q^3 \qquad \text{is unimodal}$$
$$-1 + 2q - 3q^2 + 2q^3 - q^4 \qquad \text{is symmetric}$$

Surprising Unimodal Polynomials

Example 3

All of the following functions are not only polynomial for $n \geq 0, 0, 0, 4, 4$ respectively, but they are also unimodal.

$$P_1(n) = \frac{1 - q^{n+1}}{1 - q}$$

$$P_2(n) = \frac{1 - q^{2n+1}}{1 - q}$$

$$P_3(n) = \frac{2q^{3n+2} - 2q^{3n+1} + q^{3n} - q^{2n+1} - q^{n+1} + q^2 - 2q + 2}{(1 - q)^2}$$

$$P_4(n) = \left(5q^{4n+2} - 5q^{4n+1} + 3q^{4n-2} - 3q^{4n-3} + 4q^{4n-4} - 4q^{2n+3} - 4q^{2n-1} + 4q^6 - 3q^5 + 3q^4 - 5q + 5\right) \Big/ (1 - q)^2$$

$$\begin{aligned} P_5(n) = \Big(& 2q^{5n+3} - 4q^{5n+2} + 7q^{5n+1} - 5q^{5n} + 5q^{5n-3} - 5q^{5n-4} + 4q^{5n-5} - 4q^{5n-6} + 6q^{5n-9} - 6q^{5n-10} + 3q^{5n-11} \\ & - 5q^{4n+2} + 5q^{4n+1} - 4q^{4n-2} + 4q^{4n-3} - 3q^{4n-4} - 5q^{3n+4} + 5q^{3n+3} - 3q^{3n+2} - 6q^{3n-4} + 6q^{3n-5} - 3q^{3n-6} \\ & + 3q^{2n+9} - 6q^{2n+8} + 6q^{2n+7} + 3q^{2n+1} - 5q^{2n} + 5q^{2n-1} + 3q^{n+7} - 4q^{n+6} + 4q^{n+5} - 5q^{n+2} + 5q^{n+1} \\ & - 3q^{14} + 6q^{13} - 6q^{12} + 4q^9 - 4q^8 + 5q^7 - 5q^6 + 5q^3 - 7q^2 + 4q - 2\Big) \Big/ (-1 + q)^3 \end{aligned}$$

Example 4

The following functions are all unimodal for $n \geq 0$.

$$Q_1(n) = \frac{4\left(1-q^{n+1}\right)}{1-q}$$

$$Q_2(n) = 5\,\frac{8q^{2n+4} - 2q^{n+3} - 2q^{n+2} + 8q - 3[(q^5+1)(q^n-(-q)^n) + (q^4+q)(q^n+(-q)^n)]}{2q\,(1-q)\,(1-q^2)}$$

$$\begin{aligned}
Q_3(n) = \Big(&16\big[3 - 3q + 16q^2 - 16q^{n+1} - 16q^{n+3} - 16q^{n+5} + 16q^{2n+3} \\
&\quad + 16q^{2n+5} + 16q^{2n+7} - 16q^{3n+6} + 3q^{3n+7} - 3q^{3n+8}\big] \\
&+ (1-(-1)^n)(-q)^{(3n-9)/2}\big[8q^{17} - 8q^{16} + 64q^{15} - 9q^{14} - 55q^{13} - 60q^{11} + 57q^{10} - 125q^9 \\
&\quad + 125q^8 - 57q^7 + 60q^6 + 55q^4 + 9q^3 - 64q^2 + 8q - 8\big] \\
&+ (1-(-1)^n)q^{(3n-9)/2}\big[8q^{17} - 8q^{16} + 64q^{15} + 9q^{14} - 73q^{13} - 60q^{11} - 65q^{10} - 3q^9 \\
&\quad + 3q^8 + 65q^7 + 60q^6 + 73q^4 - 9q^3 - 64q^2 + 8q - 8\big] \\
&+ (1+(-1)^n)(-q)^{(3n-6)/2}\big[12q^{14} - 12q^{13} + 64q^{12} - 75q^{11} + 74q^{10} - 127q^9 \\
&\quad + 127q^5 - 74q^4 + 75q^3 - 64q^2 + 12q - 12\big] \\
&+ (1+(-1)^n)q^{(3n-6)/2}\big[12q^{14} - 12q^{13} + 64q^{12} - 53q^{11} - 74q^{10} - q^9 \\
&\quad + q^5 + 74q^4 + 53q^3 - 64q^2 + 12q - 12\big]\Big) \Big/ 4(1-q)^2(1-q^6)
\end{aligned}$$

Why Unimodal is Useful

- Knowing a sequence is unimodal allows for easy search and guaranteed discovery of the global extremum.
- Identifying a probability distribution as unimodal allows certain approximations for how far a value will be from its
  - mode (Gauss' inequality [Gau23]) or
  - mean (Vysochanskij-Petunin inequality [DFV80]).

More Definitions

Definition 5 (Real-rootedness)

The generating polynomial, $p_{\mathcal{A}}(x) := a_0 + a_1 q + \cdots + a_n q^n$, is called *real-rooted* if all its zeros are real. By convention constant polynomials are considered to be real-rooted.

Definition 6 (Log-concavity)

A sequence $\mathcal{A} = \{a_0, \ldots, a_n\}$ is *log-concave* if $a_i^2 \geq a_{i-1}a_{i+1}$ for all $1 \leq i < n$.

Lemma 7 (Brändén [Brä15])

Let $\mathcal{A} = \{a_k\}_{k=0}^{n}$ be a finite sequence of nonnegative numbers.

- *If $p_{\mathcal{A}}(x)$ is real-rooted, then $\mathcal{A}$ is log-concave.*
- *If $\mathcal{A}$ is log-concave and positive, then $\mathcal{A}$ is unimodal.*

q-Binomial Polynomials

Definition 8 (q-Binomial Polynomial)

$$G(n,k) = \begin{bmatrix} n+k \\ k \end{bmatrix}_q = \frac{(1-q^{n+1})\cdots(1-q^{n+k})}{(1-q)\cdots(1-q^k)}$$

Example 9

$$G(2,1) = 1 + q + q^2$$
$$G(2,2) = 1 + q + 2q^2 + q^3 + q^4$$
$$\begin{aligned} G(5,3) = {} & q^{15} + q^{14} + 2q^{13} + 3q^{12} + 4q^{11} + 5q^{10} + 6q^9 + 6q^8 \\ & + 6q^7 + 6q^6 + 5q^5 + 4q^4 + 3q^3 + 2q^2 + q + 1 \end{aligned}$$

These polynomials fall into the category of unimodal, but (typically) not log-concave.

History

1. q-binomial coefficients were assumed to be unimodal as early as the 1850's.
2. 1878: Sylvester proved this claim using invariant theory.
3. 1982: Proctor gave an "elementary" proof using linear algebra.
4. 1989: Kathy O'Hara provided a **combinatorial** proof of the unimodal nature of the q-binomial coefficients.

Work

Definition 10 (Partition)

A *partition* of k is a non-increasing sequence of positive integers $\lambda = [a_1, a_2, \ldots, a_s]$ s.t. $\sum_{i=1}^{s} a_i = k$.

Proving $G(n,k)$ has symmetric coefficients is relatively easy.

1. $G(n,k) = q^k G(n-1,k) + G(n,k-1)$.
2. The ℓ^{th} coefficient of $G(n,k)$ is the number of ways a partition of ℓ could fit inside an $n \times k$ box.
3. Each partition of ℓ in an $n \times k$ box corresponds to a partition of $nk - \ell$. Look at the empty squares.

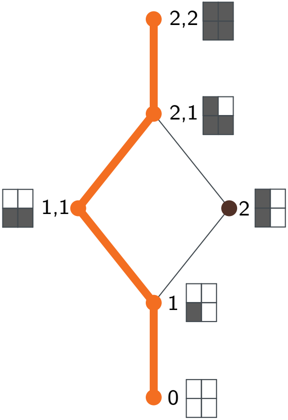
Work

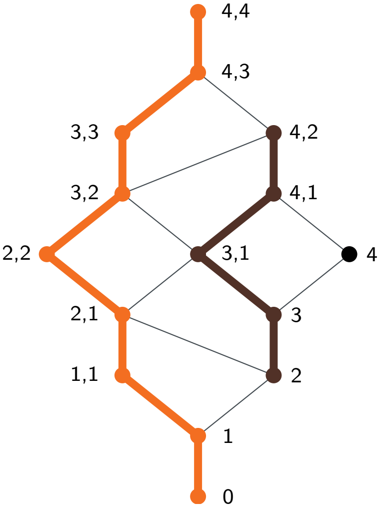
Proving unimodal is much harder.

1. As previously mentioned, we cannot use log-concave.
2. O'Hara showed unimodality using symmetric chain decomposition.
3. Dr. Z. translated the argument into an elegant recurrence.

I introduce several perturbations to the recurrence to create a larger family of unimodal polynomials.

Symmetric Chain Decomposition





Definitions

Definition 11 (Darga)

The *darga* of a polynomial $p(q) = a_i q^i + \cdots + a_j q^j$, with $a_i \neq 0 \neq a_j$, is defined to be $i + j$, i.e. the sum of its lowest and highest powers.

Example 12

$darga(q^2 + 3q^3) = 5$ and $darga(q^2) = 4$.

Propositions

Proposition 13

The sum of two symmetric and unimodal polynomials of darga m is also symmetric and unimodal of darga m.

Proposition 14

The product of two symmetric and unimodal nonnegative polynomials of darga m and m' is a symmetric and unimodal polynomial of darga $m + m'$.

Proposition 15

If p is symmetric and unimodal of darga m, then $q^{\alpha}p$ is symmetric and unimodal of darga $m + 2\alpha$.

Amazing Recurrence!

Theorem 16

$$G(n,k) = \sum_{(d_1,\dots,d_k);\sum_{i=1}^k id_i=k} q^{k\left(\sum_{i=1}^k d_i\right)-k-\sum_{1\le j<i\le k}(i-j)d_id_j} \prod_{i=0}^{k-1} G\left((k-i)n-2i+2\sum_{j=0}^{i-1}(i-j)d_{k-j},\, d_{k-i}\right) \tag{1.1}$$

The outside sum is over all partitions of k. d_i = the number of parts of size i. Initial conditions:

$$G(n<0,k)=0, \qquad G(n,k<0)=0, \qquad G(0,k)=1,$$

$$G(n,0)=1, \qquad G(n,1)=\frac{1-q^{n+1}}{1-q}.$$

Why Does It Matter?

What is so great about this recurrence?

- For any fixed value of k, it provides a one-line high-school algebra proof that $G(n, k)$ is symmetric and unimodal. [Zei89]
- If you work through some simple algebra, you can prove $G(n, k)$ is symmetric and unimodal for ALL k.
- We can slightly tweak this recurrence and still create a symmetric and unimodal polynomial.

Tweaked Recurrence Proof

Example 17

To prove the polynomials in Example 3 are actually unimodal, show that

$$P_1(n) = \sum_{i=0}^{n} q^i$$

$$P_2(n) = P_1(2n)$$

$$P_3(n) = 2P_1(3n) + q^2 P_1(2n-2) P_1(n-2)$$

$$P_4(n) = 5P_1(4n) + 3q^4 P_2(2n-4) + 4q^6 P_1(2n-4) P_2(n-4)$$

$$\begin{aligned} P_5(n) = {} & 2P_1(5n) + 5q^2 P_1(4n-2) P_1(n-2) + 4q^8 P_2(2n-6) P_1(n-4) \\ & + 5q^6 P_1(3n-4) P_2(n-4) + 3q^{12} P_1(2n-6) P_3(n-6) \end{aligned}$$

Tweaked Recurrence Proof

Example 18

To prove the polynomials in Example 4 are actually unimodal, show that

$$Q_1(n) = 4\sum_{i=0}^{n} q^i$$

$$Q_2(n) = q^2 Q_2(n-2) + 5Q_1(2n)$$

$$Q_3(n) = q^6 Q_3(n-4) + 3Q_1(3n) + 4q^2 Q_1(2n-2)\,Q_1(n-2)$$

The previous examples were generated by multiplying each term in the recurrence equation (1.1) by a random constant.

Depth

Lemma 19

If $k = 1, 2$ or if $k \geq 4$ is even (odd) and $n \geq 2$ ($n \geq 4$), then the maximum depth of recursive calls is

$$\left\lceil \frac{\lfloor k/2 \rfloor n}{2} \right\rceil - \left\lceil \frac{k}{2} \right\rceil + 1.$$

For $k = 3$, the maximum depth of recursive calls is $\left\lceil \frac{n}{4} \right\rceil$.

The important take away is that the recurrence is still linear in computation depth.

Bounded Partition Parts

Lemma 20

Suppose we restrict partitions to only use integers $\leq p$. If $p < \frac{2k}{n+2}$, then $G'(n,k) = 0$.

So restricting the size of parts (at least to constant height) is not very interesting.

Restricted Partition Size

Let G_s denote the polynomials obtained from restricting the recurrence to partitions of size $\leq s$: $\sum_{i=1}^{k} d_i \leq s$.

Conjecture 21

$$(q^n - q^{k+s-1}) G_s(n+1, k+1) = q^{k+1}(q^n - q^{s-2}) G_s(n, k+1) + q^n(1 - q^{k+s}) G_s(n+1, k) \tag{1.2}$$

Eqn. (1.2) was verified for $n, k \leq 20$ and $s \leq 10$.
Note that $G_s(n, k) = G(n, k)$ for $k \leq s$. If we take $s \to \infty$, then we obtain the "simple" recurrence given earlier.

Particular Recursive Calls

What does each recursive call look like? Eqn. (1.1) is a summation over partitions so I chose to look at what some special partitions contribute.

If $\lambda = [k]$, then the power of q in front is q^0 and the product call is to $G(nk, 1) = \frac{1-q^{nk+1}}{1-q}$.
Thus $[k]$ provides the "base" of $G(n, k)$:
$1 + q + \cdots + q^{nk}$.

If we restrict to partitions of size 1, then $G_1(n, k) = \frac{1-q^{nk+1}}{1-q}$.
We can use this to verify the recurrence relation in Eqn. (1.2) for $s = 1$.

Particular Recursive Calls

Then I considered $\lambda = [k-\ell, \ell]$ for some $1 \leq \ell < \frac{k}{2}$. I found a recursive call of

$$q^{2\ell} G((k-\ell)n - 2\ell, 1) G(\ell n - 2\ell, 1) = q^{2\ell} \frac{1 - q^{(k-\ell)n - 2\ell + 1}}{1-q} \cdot \frac{1 - q^{\ell n - 2\ell + 1}}{1-q}.$$

For the special case $\lambda = \left[\frac{k}{2}, \frac{k}{2}\right]$ when k is even, the recursive call is

$$q^k G\left(\frac{k}{2}n - k, 2\right)$$

Maximum Size 2

We now have all of the contributions from partitions of size ≤ 2.

Lemma 22

$$G_2(n,k) = \frac{1-q^{nk+1}}{1-q} + q^2\frac{(1+q^{nk-k+1})(1-q^{k-1})}{(1-q)^2(1-q^2)} - q^{n+1}\frac{1-q^{n(k-1)}}{(1-q)^2(1-q^n)}.$$

We can now verify the recurrence relation in Eqn. (1.2) for $s = 2$.

Maximum Size 3

We could explicitly construct $G_3(n, k)$ by looking at the contributions of partitions $\lambda = [\ell_1, \ell_2, \ell_3]$ for the separate cases

1. $\ell_1 = \ell_2 = \ell_3$,
2. $\ell_1 = \ell_2 \neq \ell_3$,
3. $\ell_1 \neq \ell_2 = \ell_3$,
4. and $\ell_1 \neq \ell_2 \neq \ell_3$.

Then we could verify the recurrence relation in Eqn. (1.2) for $s = 3$.

Maximum Size 3

Lemma 23

$$
\begin{aligned}
G_3(n,k) = \Big[& q^{4k} \cdot q^3 (1-q^n)(q-q^n) - q^{3k} \cdot q(1+q)(1-q^n)\left(q - q^2 + q^5 - q^n\right) \\
& - q^{2k}\Big(q^{nk}\Big(q^{2n}(q^9 - q^8 - q^7 + q^6 + q^5 - q^3 + q) \\
& \qquad - q^n(q^{10} - q^8 + q^6 + q^5) + q^{10}\Big) \\
& \qquad - q^{2n} + q^n(q^5 + q^4 - q^2 + 1) \\
& \qquad - q^9 + q^7 - q^5 - q^4 + q^3 + q^2 - q\Big) \\
& + q^k \cdot q^{nk} q^3 (1+q)(1-q^n)\left(q^5 - q^n + q^{n+3} - q^{n+4}\right) \\
& - q^{nk} q^6 (1-q^n)(q-q^n)\Big] \\
& \Big/ (1-q)^2(1-q^2)^2(1-q^3)(1-q^{n-1})(1-q^n)q^{2k+1}
\end{aligned}
$$

Maximum Size Beyond 3

This would not work for proving true for every s: an infinite number of cases to handle!

Ideally, there is a way to interpret the recurrence combinatorially. Recall

$$G(n,k) = q^k G(n-1,k) + G(n,k-1)$$

Then, if we are very lucky, it is possible to describe the number of new objects in closed form.

Other Adjustments

There are many other ways to modify the recurrence and still obtain unimodal polynomials.

- As shown earlier, we can simply multiply the recursive calls by constants.
- We can adjust the initial conditions.
- A more complicated method is to adjust the recursive call.

Adjusted Recursive Call

$$G\left((k-i)n-2i+2\sum_{j=0}^{i-1}(i-j)d_{k-j}-2a(k-i)-2b,\ d_{k-i}-2c\right)$$

Proposition 24

1. *$G(n,k;a,b)$ will have smallest degree $ka+b$.*
2. *If $a+b>\frac{n}{2}$, then $G(n,k;a,b)=0$.*
3. *If $a+b=\frac{n}{2}$, then $G(n,k;a,b)=q^{ka+b}\frac{1-q^{nk+1-2(ka+b)}}{1-q}$.*

Conclusion

- Building an interesting combinatorial object from scratch can be difficult.
- Taking one and tweaking its construction is much easier.
- Computers are very helpful.
- There is still more to analyze about this recurrence.
- Ultimate goal: Find restrictions on the partitions/other parameters that yield closed-form solutions in 2 (or more!) variables.

Future Work

- Arbitrarily combine polynomials of known darga to create another polynomial with known darga. Do this recursively in a similar manner.
- Instead of looking at partitions, look at something else?
- Is factoring a symmetric/unimodal polynomial any easier than generic?
- Use any of these polynomials as a probability distribution.

Abstract

We employ a generating function relation technique used by Ayyer and Zeilberger to analyze lattice walks with a general step set in bounded, semi-bounded, and unbounded planes.
The method in which we do this is formulated to be highly algorithmic so that a computer can automate most, if not all, of the work.

Introduction

We consider walks in the two-dimensional square lattice with an "arbitrary" set of integral steps (x, y) subject to $x \geq 0$.

We want to count all possible walks of a certain length. Rather than a brute-force search of the entire space, looking for 1 value, we will use generating function relations. As a bonus, we obtain not only the initial generating function of desire, but also many related ones of interest.

Motivation

An earlier motivation for bounded walks came from Physics: analyzing polymers constrained between plates [BORW05]. Zeilberger gave one solution in an earlier paper [AZ07] that provided the main motivation for this research.

The kernel method has received attention lately for analyzing specific cases of walks. Compared to the kernel method, we believe our method is a lot easier to understand combinatorially, is more insightful, faster, and easier to produce.

Trying to picture an entire walk at once can be difficult. This is where the awesome powers of dynamical programming come into play.

Definitions

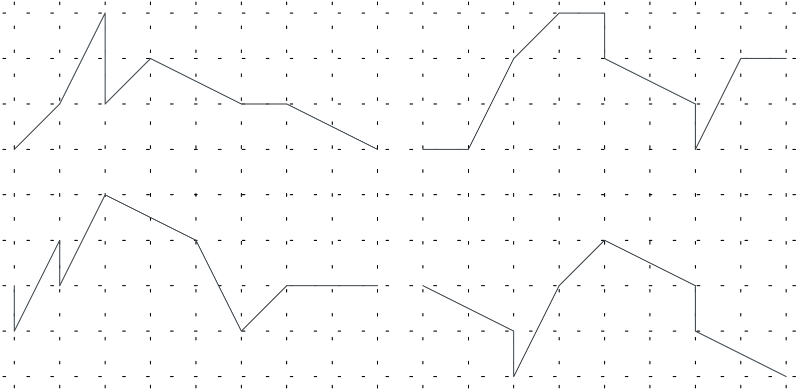
I will generically use the term *walk* to indicate any sequence of points $\{(x_0, y_0), \dots, (x_s, y_s)\}$ in the xy-plane. The steps of the walk are then
$\{(x_1 - x_0, y_1 - y_0), \dots, (x_s - x_{s-1}, y_s - y_{s-1})\}$.

Definition 25 (Walks)

A *bridge* is an unbounded walk that begins at the origin and ends on the x-axis. I say *bounded bridge* for a bounded walk that begins at the origin and ends on the x-axis.
An *excursion* is a semi-bounded walk that begins at the origin and ends on the x-axis.
A *free* walk can end anywhere. A *meander* is a semi-bounded free walk.
The *length* of a walk is $n = \sum_{i=0}^{s} x_i$. The *size* of a walk is s.

Examples

Figure: Walk Examples



$$\mathcal{S} = \{[0,-1],[1,0],[1,1],[1,2],[2,-1]\}.$$

Irreducible Walks

I will refer to the y-value as the *altitude* of the walk.

Definition 26

The *interior* of a walk consists of every point other than the endpoints: $\{(x_1, y_1), \ldots, (x_{s-1}, y_{s-1})\}$.
An *irreducible* walk is one in which the interior has a strictly higher altitude than the lower endpoint: $\min\{y_1, \ldots, y_{s-1}\} > \min\{y_0, y_s\}$.

Irreducible is also sometimes used to refer to walks that do not hit the final altitude until the final step.

Bounded Walks

The goal is to find the g.f., denoted $f_{a,b}$, for walks with step set $\mathcal{S}$, starting at $(0,0)$, and bounded above and below by $a \geq 0$ and $b \leq 0$ respectively.

First assume that the walk is free. Then

$$f_{a,b} = 1 + \sum_{(x,y)\in\mathcal{S}} t^x f_{a-y,b-y}$$

So we can describe $f_{a,b} \ldots$ using other $f_{a',b'}$.

Bounded Walks

$$f_{0,b-a} = 1 + \sum_{(x,y)\in\mathcal{S}} t^x f_{0-y,b-a-y}$$

$$f_{1,b-a+1} = 1 + \sum_{(x,y)\in\mathcal{S}} t^x f_{1-y,b-a+1-y}$$

$$\vdots$$

$$f_{a-b,0} = 1 + \sum_{(x,y)\in\mathcal{S}} t^x f_{a-b-y,0-y}.$$

This is a (linear!) system of $a-b$ equations with $a-b$ variables.

Example

Example 27 (Close (American) Football Games)

$$\mathcal{S} = \{[1,2],[1,3],[1,6],[1,7],[1,8],[1,5],[1,4],\\ [1,-2],[1,-3],[1,-6],[1,-7],[1,-8],[1,-5],[1,-4]\}.$$

Bound by $y = 8, -8$. Then the g.f. is explicitly

$$\frac{1+10t+13t^2-37t^3-40t^4+28t^5+26t^6-2t^7}{1-4t-59t^2-77t^3+170t^4+234t^5-92t^6-142t^7-4t^8+6t^9}.$$

This sequence is new in the OEIS: A301379[OEIa].

Speed Enumeration

Table: Bounded Walk Enumeration 1000 terms

Method	Memory	CPU Time	Real Time
Brute-Force Recursion	60.23MiB	309.53ms	308.90ms
G.F. Construction	3.27MiB	30.17ms	33.50ms
Taylor Enumeration	4.86MiB	4.93ms	5.07ms
Total	8.13MiB	35.10ms	38.57ms

Using the g.f. method of enumeration is about 8 times as fast and uses a much smaller amount of memory: (1/8th).

Bounded Bridges

We have solved for walks that end ANYWHERE. Let $f_{a,b}$ now denote the g.f. for walks that begin at $(0,0)$ and end on the x-axis.
Let $e_{a,b,c}$ denote the g.f. for paths that start at $(0,c)$, end on the x-axis and never touch the x-axis beforehand.

$$f_{a,b} = 1 + \left(\sum_{(x,0)\in\mathcal{S}} t^x \right) f_{a,b} + \left(\sum_{(x,y)\in\mathcal{S};y\neq 0} t^x e_{a,b,y} \right) f_{a,b} \quad (2.1)$$

We now need the equations for $e_{a,b,c}$ for $a \geq c \geq b$.

$$e_{a,b,c} = \sum_{(x,-c)\in\mathcal{S}} t^x + \sum_{(x,y)\in\mathcal{S};y\neq -c} t^x e_{a,b,c+y}$$

Example

Example 28 (Tied (American) Football Games)

$$\mathcal{S} = \{[1,2],[1,3],[1,6],[1,7],[1,8],[1,5],[1,4],\\ [1,-2],[1,-3],[1,-6],[1,-7],[1,-8],[1,-5],[1,-4]\}.$$

Bound by $y = 8, -8$. Then the g.f. is explicitly

$$\frac{1-4t-45t^2-43t^3+98t^4+108t^5-24t^6-30t^7}{1-4t-59t^2-77t^3+170t^4+234t^5-92t^6-142t^7-4t^8+6t^9}.$$

This sequence is new in the OEIS: A301380[OEIb].

Speed Enumeration

Table: Bounded Bridge Enumeration

Method	Memory	CPU Time	Real Time
Brute-Force Recursion	55.98MiB	278.93ms	279.03ms
G.F. Construction	10.13MiB	102.43ms	103.20ms
Taylor Enumeration	4.85MiB	5.20ms	5.10ms
Total	14.98MiB	107.63ms	108.30ms

Again, the g.f. method is faster; this time, about 2.5 times as fast and 1/4th the memory.

Excursions

Let f now denote the g.f. for nonnegative excursions. Then we can try:

$$f = 1 + \left(\sum_{(x,0) \in \mathcal{S}} t^x \right) f + \left(\sum_{(x,y) \in \mathcal{S}; y \neq 0} t^x e_y \right) f$$

Better Excursion Method

Let $g_{a,b}$ denote the g.f. for irreducible walks from $(0, a)$ to (n, b). Now we have

$$f_{0,0} = 1 + \left(g_{0,0} + \sum_{(x,0)\in\mathcal{S}} t^x \right) f_{0,0} \tag{2.2}$$

and

$$g_{0,0} = \sum_{(x_1,y_1)\in\mathcal{S};y_1>0} \; \sum_{(x_2,y_2)\in\mathcal{S};y_2<0} t^{x_1} f_{y_1-1,-y_2-1} t^{x_2}. \tag{2.3}$$

Better Excursion Method

$$a > b \qquad f_{a,b} = \sum_{i=0}^{b} g_{a-i,0} f_{0,b-i}, \tag{2.4}$$

$$a = b \qquad f_{a,a} = \sum_{i=0}^{b-1} g_{a-i,0} f_{0,b-i} + f_{0,0}, \tag{2.5}$$

$$a < b \qquad f_{a,b} = \sum_{i=0}^{a-1} g_{a-i,0} f_{0,b-i} + f_{0,0} g_{0,b-a}. \tag{2.6}$$

$$g_{a,0} = \sum_{(x,y)\in\mathcal{S};y<0} f_{a-1,-y-1} t^x, \tag{2.7}$$

$$g_{0,a} = \sum_{(x,y)\in\mathcal{S};y>0} t^x f_{y-1,a-1}. \tag{2.8}$$

Example $\mathcal{S} = \{[0,-2],[2,0],[1,1],[3,3]\}$

$$f_{0,0} = 1 + \left(t^2 + g_{0,0}\right) f_{0,0}$$

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$$f_{2,0} = g_{2,0} f_{0,0}$$

$$g_{2,0} = f_{1,1}$$

$$f_{1,1} = g_{1,0} f_{0,1} + f_{0,0}$$

$$g_{1,0} = f_{0,1}$$

$$f_{2,1} = g_{1,0} f_{0,0} + g_{2,0} f_{0,1}$$

Minimal Polynomial

Definition 29 (Minimal Polynomial)

the (minimal) algebraic equation satisfied by the generating function: p such that $p(f) = 0$ in terms of formal power series. We refer to p as the *minimal polynomial*.

So for the previous step set,

$$0 = 1 + \left(t^2 - 1\right) f + t^2 {f_{0,0}}^3 + t^4 {f_{0,0}}^4 + 2t^4 {f_{0,0}}^5 + t^6 \left(t^2 - 1\right) {f_{0,0}}^6 + t^6 \left(t^2 - 1\right)^2 {f_{0,0}}^7 + t^{10} {f_{0,0}}^9 + t^{12} {f_{0,0}}^{10}$$

Another Example

We want to obtain the number of nonnegative excursions with step set $\mathcal{S} = \{[1,-2],[1,-1],[1,0],[1,1],[1,2]\}$. Let F denote the corresponding g.f.; then

$$t^4F^4 - t^2(t+1)F^3 + t(t+2)F^2 - (t+1)F + 1 = 0. \quad (2.9)$$

A truncated solution in formal power series, and the one that makes sense in terms of our problem, is

$$F = 1+t+3t^2+9t^3+32t^4+120t^5+473t^6+1925t^7+8034t^8+\cdots$$

Speed Excursion Enumeration

Table: 500 term Excursion Enumeration

Method	Memory	CPU Time	Real Time
Vector Set-Up	16.58KiB	300μs	766μs
Iterating	100.04GiB	4.45m	4.09m
Total	100.04GiB	4.45m	4.09m
Polynomial	2.63MiB	19.23ms	18.63ms
Iterating	23.61GiB	64.52s	58.56s
Total	23.61GiB	64.54s	58.58s
Polynomial	2.63MiB	19.23ms	18.63ms
taylor	328.97MiB	3.20s	3.18s
Total	331.60MiB	3.22s	3.20s
Brute-Force Recursion	186.60MiB	1.569s	1.472s

Meanders

What if we don't care where the walk ends, as long as it stays above $y = -c$?
Let k_a denote the g.f. for meanders that begin at $(0, a)$, restricted to step set $\mathcal{S}$. Then

$$k_0 = 1 + \left(g_{0,0} + \sum_{(x,0)\in\mathcal{S}} t^x \right) k_0 + \sum_{(x,y)\in\mathcal{S};y>0} t^x k_{y-1}$$

We need only describe the new k_i.

$$k_a = \sum_{i=0}^{a-1} g_{a,i} k_0 + k_0 = \left(\sum_{i=1}^{a} g_{i,0} + 1 \right) k_0.$$

Meander Example

Let's use $\mathcal{S} = \{[1,-2],[1,-1],[1,0],[1,1],[1,2]\}$. The g.f., K, for the number of nonnegative meanders satisfies

$$t^2(5t-1)^2K^4 + t(5t-1)^2K^3 + 3t(5t-1)K^2 + (5t-1)K + 1 = 0.$$

and has truncated solution

$$K = 1 + 3t + 12t^2 + 51t^3 + 226t^4 + 1025t^5 + 4724t^6 + 22022t^7$$

Enumerating was roughly equivalent.

Guessing

Since we know we will obtain a minimal polynomial, why not just guess?

Table: Finding Excursion Minimal Polynomial
$\mathcal{S} = \{[1,-2],[1,-1],[1,0],[1,1],[1,2]\}$

Method	Memory Used	CPU Time	Real Time
New Method	2.63MiB	19.23ms	18.63ms
Empir	91.61MiB	734ms	735ms
EmpirF	4.33MiB	33.87ms	35.90ms

Better Recursion

There is a classical method for deducing from the algebraic function satisfying the g.f. a linear recurrence with polynomial coefficients satisfied by the coefficients of the g.f. in question.[KP11].

Interestingly, sometimes a larger (non-minimal) polynomial produces a better (lower-order) recurrence.

Better Recursion Example

Let $\mathcal{S} = \{[1,-2],[1,-1],[1,0],[1,1],[1,2]\}$. Let F denote the g.f. for nonnegative excursions. Then its coefficients satisfy

$$
\begin{aligned}
0 = {} & 3125(n+1)(n+2)(n+3)(n+4)B(n) \\
& - 250(n+4)(n+3)(n+2)(27n+122)B(n+1) \\
& + 25(n+4)(n+3)(107n^2+1457n+4316)B(n+2) \\
& + 10(n+4)(304n^3+3233n^2+9864n+6513)B(n+3) \\
- & (2821n^4+56794n^3+425771n^2+1407974n-1731540)B(n+4) \\
& + 2(n+7)(413n^3+6986n^2+39356n+73830)B(n+5) \\
& - (n+8)(n+7)(99n^2+1241n+3900)B(n+6) \\
& + 2(2n+15)(n+9)(n+8)(n+7)B(n+7). \qquad (2.10)
\end{aligned}
$$

Enumeration Speed

Method	Memory Used	CPU Time	Real Time
Polynomial	2.63MiB	19.23ms	18.63ms
algtorec	35.84MiB	267.93ms	262.40ms
SeqFromRec	123.05MiB	375.27ms	375.37ms
Total	161.52MiB	662.43ms	656.40ms

This was for 1000 terms. Brute-force took 1.5 seconds for only 500 terms.

Guessing Again

Again, since we know we can find a linear recurrence, why don't we just guess?

There are cases ($\mathcal{S} = \{[1, 2], [1, -3]\}$) where guessing is significantly faster, but there are also cases ($\mathcal{S} = \{[1, -1], [3, -1], [1, 0], [3, 0], [2, 1], [1, 2], [2, 2]\}$) when converting is much faster.

The key for deciding is the degree of the minimal polynomial, $|\mathcal{S}|$, and the size of the largest step.

Free Walk

This is not very interesting. For a step set $\mathcal{S}$, the g.f. is explicitly

$$\frac{1}{1 - \sum_{(x,y)\in\mathcal{S}} t^x}.$$

The actual altitude doesn't matter.

Bridges

A lot of the work we have done in the semi-bounded case will prove useful here. We cannot use the exact same method as for excursions.

Suppose we tried describing a walk with a negative change in altitude. The first and last steps could both be positive. Then we would need to describe a walk that has a larger negative change in altitude. And so on.

Bridges

Let G denote the g.f. of bridges with a step set $\mathcal{S}$.

$$G = 1 + \left(h_0 + \sum_{(x,0)\in\mathcal{S}} t^x \right) G.$$

We now have a new type of "irreducible" g.f.

$$h_0 = 2g_{0,0} + \sum_{(x,y)\in\mathcal{S};y\leq -2} \sum_{i=1}^{-y-1} g_{0,i} t^x h_{i+y} + \sum_{(x,y)\in\mathcal{S};y\geq 2} \sum_{i=1}^{y-1} g_{y-i,0} t^x h_i$$

Irreducible Unbounded Walks

Note that walks below the x-axis are in bijection with walks above the x-axis by reversing the order of steps. We can then use $g_{0,-a} = g_{a,0}$ and $f_{-a,-b} = f_{b,a}$.

$$j > 0 \qquad h_j = g_{j,0} + \sum_{(x,y)\in\mathcal{S};y\leq-2} \sum_{i=1}^{-y-1} f_{j-1,i-1} t^x h_{i+y},$$

$$j < 0 \qquad h_j = g_{0,-j} + \sum_{(x,y)\in\mathcal{S};y\geq 2} \sum_{i=1}^{y-1} f_{y-i-1,-j-1} t^x h_i.$$

Let $\mathcal{S}' = \{[2,-2],[1,-1],[1,1],[2,2]\}$ and G denote the g.f. of bridges. Then G satisfies

$$\begin{aligned} 0 = 1 + 2\left(4t^2 - 3\right)\left(4t^4 - 8t^2 + 1\right) G^2 \\ \left(8t^2 + 5\right)\left(4t^4 - 8t^2 + 1\right)^2 G^4 \end{aligned}$$

and has truncated solution

$$G = 1 + 2t^2 + 14t^4 + 84t^6 + 556t^8 + 3736t^{10}.$$

Combining Solutions

Sports statistics are an integral part of many a fan base. So a question of interest beyond the number of ways to be tied, may be the number of ways to win by at least X. How do we describe this walk?

Suppose we want to win by ≥ 2 and never trail by more than 3. Well $f_{3,5}$ will count walks that don't drop by more than 3, and we will have at least a 2 point lead at the end. So does $f_{3,5} \cdot k_0$ count what we want? Not necessarily. $f_{3,5}$ ensures that at some point we are exactly 2 points ahead of where we began. But depending on the step set, a win may skip over this lead and never actually hit the altitude 2 steps higher than our beginning. And $f_{3,5} \cdot k_0$ double counts some walks and misses others.

Win by 2

Let L denote the g.f. of interest. Then $L = k_3 - f_{3,0} - f_{3,1} - f_{3,2} - f_{3,3} - f_{3,4}$. Let $\mathcal{S} = \{[0,-1],[1,0],[1,1],[1,2]\}$. Then L has a minimal polynomial of degree 3 and has truncated expansion.

$$L = t + 21t^2 + 305t^3 + 4064t^4 + 52431t^5 + 666657t^6 + 8420130t^7 + 106070229t^8 + 1335635352t^9.$$

Weighted Walks

We can introduce a weight to each step

$$f_{a,b} = 1 + \sum_{(x,y,w)\in\mathcal{S}} w \cdot t^x f_{a-y,b-y}.$$

We can use weights as probabilities and find the probability that a random walk is a bridge or excursion or meander.

Asymptotics

Since we can find linear recurrences for the sequences, we can find asymptotic relatively easily. Let $\mathcal{S} = \{[1,-1],[1,0],[1,2]\}$. Then,

$$0 = 1 + (t-1)F + t^3F^3,$$

$$0 = 31(n+1)(n+2)B(n) - 6(2n+5)(n+2)B(n+1)$$
$$+ 2(6n^2+36n+53)B(n+2) - 2(2n+9)(n+3)B(n+3),$$

$$B(n) \sim 0.800119 \cdot \frac{2.88988^n}{n^{3/2}} \cdot \left(1 - \frac{1.74757}{n}\right),$$

Asymptotics

$$
\begin{aligned}
0 &= 1 - 3(t-1)G + (31t^3 - 12t^2 + 12t - 4)G^3, \\
0 &= 31(n+1)(n+2)C(n) - 6(n+2)(2n+3)C(n+1) \\
&\quad + 2(6n^2 + 24n + 23)C(n+2) - 2(n+3)(2n+3)C(n+3),
\end{aligned}
$$

$$C(n) \sim 0.3488332 \cdot \frac{2.88988^n}{\sqrt{n}} \cdot \left(1 - \frac{0.247572}{n}\right).$$

The proportion of excursions to bridges is

$$\frac{B(n)}{C(n)} \sim \frac{2.293700526}{n}.$$

Meander Asymptotic

Let $\mathcal{S} = \{[1,-1],[1,0],[1,1],[2,2]\}$. Then

$$0 = 1 + (3t^2+3t-1)K + t(3t+1)(t^2+3t-1)K^2 + t^2(t^2+3t-1)^2K^3,$$

$$D(n) \sim \frac{136}{443}\left(\frac{3+\sqrt{13}}{2}\right)^n \approx 0.307 \cdot 3.303^n$$

Conclusion

We analyzed walks by looking at certain steps along its route.

- The first step for bounded walks.
- The last step for semi-bounded walks.
- A middle step for unbounded walks.

We always created a CLOSED system of relations!

Future Work

One avenue that I am pursuing is counting the area under each walk. This involves adding a catalytic variable z that counts that area at each step.

The result is no longer “solvable”. But we can iterate for a solution.

Conclusion

Computers were instrumental in my being able to research. I never would have seen the depth formula or conjectured recurrence for unimodal polynomials. And producing lattice walk g.f.s would have been extremely tedious. Once I had the algorithm down, a computer could easily show me that these methods are accurate.

I revised results of mathematicians before me to work in more ways than originally intended. In the process I created programs for others to work further.

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Questions?

Unimodal Polynomials
Lattice Walk Enumeration
Conclusion

Questions?

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