Solutions to Dr. Z.'s Math 421(2) REAL Quiz #9

1. (5 points) Solve the boundary value pde problem:

$$u_{xx} = u_{tt} \quad , 0 < x < \pi \quad , \quad t > 0 \quad ;$$

$$u(0,t) = 0 \quad , \quad u(\pi,t) = 0 \quad , \quad t > 0 \quad ;$$

$$u(x,0) = 3\sin x \quad , \quad u_t(x,0) = 6\sin 3x \quad , \quad 0 < x < \pi \quad .$$

Sol. Using Dr. Z.'s shortcut method, the answer is

$$u(x,t) = 3\sin x \cos t + \frac{6\sin(3x)\sin(3t)}{3} = 3\sin x \cos t + 2\sin(3x)\sin(3t) \quad .$$

2. (5 points) Solve :

$$u_{xx} + u_{yy} = 0 \quad , \quad 0 < x < \pi \quad , \quad 0 < y < 1 \quad ,$$

Subject to

$$u(0,y) = 0 \quad , \quad u(\pi,y) = 0 \quad , \quad 0 < y < 1 \quad ;$$

$$u(x,0) = 0 \quad , \quad u(x,1) = 3 \quad , \quad 0 < x < \pi \quad .$$

Sol. There are eight families of product solutions

$$\cos(\lambda x)\cosh(\lambda y) \quad , \quad \cos(\lambda x)\sinh(\lambda y) \quad , \quad \sin(\lambda x)\cosh(\lambda y) \quad , \quad \sin(\lambda x)\sinh(\lambda y) \quad ,$$

and four others obtained by exchanging cos and cosh and sin and sinh. To make $u(0,y) = 0$ happy, we must rule out the first two, since plugging-in $x = 0$ does not yield 0. To male $u(x,0) = 0$ happy, we must rule out the third one, so all that remains is $u(x,y) = \sin(\lambda x)\sinh(\lambda y)$ (and its analog $u(x,y) = \sinh(\lambda x)\sin(\lambda y)$). To make $u(\pi,y) = 0$ we need $u(\pi,y) = \sin(\lambda\pi)\sinh(\lambda y) = 0$. Since $\sinh(\lambda y)$ is not the zero function, we must insist that $\sin(\lambda\pi) = 0$ (for the analog $\sinh(\lambda\pi) = 0$, but this never happens, so we can forget about that option).

Solving the trig equation $\sin(\lambda\pi) = 0$ yields $\lambda\pi = n\pi$, solving for λ gives $\lambda = n$, n integer.

So the inifinte family $\{(\sin nx)(\sinh ny)\}$ are all solutions of the pde plus the first three conditions. By the superposition principle so is *any* combination

$$u(x,y) = \sum_{n=1}^{\infty} A_n(\sin nx)(\sinh ny) \quad ,$$

for **any** numbers $A_1, A_2, A_3, \ldots$.

It remains to make the last condition happy $u(x,1)=3$. Plugging-in $y=1$:

$$u(x,1)=\sum_{n=1}^{\infty}A_n(\sin nx)(\sinh n)=\sum_{n=1}^{\infty}(A_n\sinh n)(\sin nx)$$

So

$$3=\sum_{n=1}^{\infty}(A_n\sinh n)(\sin nx) \quad .$$

So $A_n\sinh n$ are the **Fourier-Sine** coefficients of 3.

We have to find the **Fourier-Sine** series of 3 (no shortcuts are possible!) By the formula

$$f(x)=\sum_{n=1}^{\infty}a_n\sin nx$$

with

$$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx\,dx \quad ,$$

we get

$$a_n=\frac{2}{\pi}\int_0^{\pi}3\sin nx\,dx=\frac{6}{\pi}\int_0^{\pi}\sin nx\,dx=\left(\frac{6}{\pi}\right)\cdot\frac{(-\cos nx)}{n}\Big|_0^{\pi}=\frac{-6}{n\pi}(\cos 0-\cos n\pi)=\frac{-6}{n\pi}(1-(-1)^n) \quad ,$$

so

$$A_n\sinh n=\frac{6}{n\pi}((-1)^n-1) \quad ,$$

and solving for A_n we get

$$A_n=\frac{6}{n\pi sinh(n)}((-1)^n-1).$$

Going back to $u(x,y)$ we get

$$u(x,y)=\frac{6}{\pi}\sum_{n=1}^{\infty}\frac{(1-(-1)^n)}{nsinh(n)}(\sin nx)(\sinh ny) \quad .$$

This is the **answer**.

Comments: 1. This was a long problem, so people who wrote

$$A_n=\frac{6}{\pi sinh(n)}\int_0^{\pi}\sin nx\,dx \quad ,$$

also got full credit.