Solutions to Dr. Z.'s Math 421(2) REAL Quiz #8

1. Solve

$$3\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad , \quad 0 < x < 2\pi \quad , \quad t > 0$$

subject to the conditions

$$u(0,t) = 0 \quad , \quad u(2\pi,t) = 0 \quad , \quad t > 0$$

$$u(x,0) = 300$$

(You may use the ready-made formula)

Sol. Note that here $L = 2\pi$ so we need the **general formula**. from the formula sheet

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k(n^2\pi^2/L^2)t} \sin\frac{n\pi}{L}x \quad ,$$

where

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx \quad .$$

First note that $k = 3$, so we have

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-3(n^2\pi^2/(2\pi)^2)t} \sin\frac{n\pi}{2\pi}x \quad .$$

Where

$$A_n = \frac{2}{2\pi}\int_0^{2\pi} f(x)\sin\frac{n\pi}{2\pi}x\,dx \quad .$$

Simplifying, using algebra:

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-3n^2t/4}\sin(nx/2) \quad .$$

Where

$$A_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx/2)\,dx \quad .$$

Since $f(x) = 300$, we have

$$A_n = \frac{1}{\pi}\int_0^{2\pi} 300\sin(nx/2)\,dx = \frac{300}{\pi}\int_0^{2\pi}\sin(nx/2)\,dx = \left(\frac{300}{\pi}\right)\left(-\frac{\cos(nx/2)}{n/2}\Big|_0^{2\pi}\right)$$

$$\left(-\frac{300}{\pi(n/2)}\right)(\cos(n(2\pi)/2) - 1) = \left(-\frac{300}{\pi(n/2)}\right)(\cos n\pi - 1) = -\frac{600((-1)^n - 1)}{\pi n} \quad .$$

Going back to $u(x,t)$ we get that the **ans.** is:

$$u(x,t) = \frac{600}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}e^{-3n^2t/4}\sin(nx/2) \quad .$$

Since $1-(-1)^n$ is 0 when n is even and is 2 when n is odd, we can repace n by $2k+1$ and have a **beter** ans.

$$u(x,t) = \frac{1200}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}e^{-3(2k+1)^2t/4}\sin((2k+1)x/2) \quad .$$