Solutions to Dr. Z.'s Math 421(2) REAL Quiz #6

1. Find the eigenfunctions and eigenvalues for the following boundary value function.

$$y'' + \lambda^2 y = 0 \quad , \quad y(0) = 0 \quad , \quad y(2\pi) = 0 \quad .$$

Sol. The **general solution** is

$$y(x) = c_1 \cos \lambda x + c_2 \sin \lambda x \quad .$$

Plugging-in $x = 0$ gives:

$$y(0) = c_1 \cos 0 + c_2 \sin 0 = c_1 \quad .$$

$y(0) = 0$ (from the data), so $c_1 = 0$. Going back to the general solution, $y(x)$, we get

$$y(x) = c_2 \sin \lambda x \quad .$$

Now we plug-in $x = 2\pi$, and get

$$y(2\pi) = c_2 \sin \lambda 2\pi \quad .$$

But, from the date of the problem, $y(2\pi) = 0$, (and $c_2 \neq 0$, or else we get the boring **zero** solution that does not count!), so the **characteristic equation** (aka as the **eigenvalue equation**) is:

$$\sin 2\pi\lambda = 0 \quad .$$

From the graph of $\sin w$ we know that $\sin w = 0$ whenever $w = 0, \pm\pi, \pm 2\pi, \ldots$, in other words $w = n\pi$ for **integer** n. So $2\pi\lambda = n\pi$ and we get that the eigenvalues are $\lambda_n = \frac{n\pi}{2\pi} = \frac{n}{2}$ (n integer). The corresponding **eigenfunction** is obtained by going back to $y(x)$ getting (put $c_2 = 1$, as is conventional)

$$y_n(x) = \sin \frac{nx}{2} \quad .$$

but when $n = 0$ we get the **zero** function, so we have to exclude it.

Ans. to 1: The eigenvalues are $\lambda_n = \frac{n}{2}$ (n integer, $n \neq 0$) and the corresponding eigenfunctions are $y_n(x) = \sin \frac{nx}{2}$.

Comment: Unfortunately only about %40 of the people got everything right. Some people had trouble solving $\sin \lambda 2\pi = 0$. Remember, the solution of the trig-equation $\sin w = 0$ is $w = n\pi$ (n integer).

Some people got the eigenvaule right $n/2$, but messed up the eignefunction $y_n(x)$. Please review it.