Solutions to Dr. Z.'s Math 421(2), (Fall 2014, RU) REAL Quiz #5

1. Find the Fourier series of $f(x) = 2x^2 + 1$ on the interval $(-\pi, \pi)$.

Note: You may use the ready-made indefinite integrals:

$$\int x^2 \sin nx = \frac{(2 - n^2x^2)\cos(nx) + 2nx\sin(nx)}{n^3}$$

$$\int x^2 \cos nx = \frac{(n^2x^2 - 2)\sin(nx) + 2xn\cos(nx)}{n^3} \quad .$$

Sol. The Fourier series of 1 is 1 and the Fourier series of $2x^2$ is twice that of x^2, so let's first find the Fourier series of x^2.

The Fourier Series, in general, is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad ,$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \quad , \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad , \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \quad .$$

Since x^2 is an **even** function, we can ignore b_n (they are automatically zero).

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{x^3}{3\pi}\Big|_{-\pi}^{\pi} = \frac{2\pi^2}{3}$$

Now, by the formula

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx = \frac{(n^2x^2 - 2)\sin(nx) + 2xn\cos(nx)}{\pi n^3}\Big|_{-\pi}^{\pi}$$

$$= \frac{(n^2\pi^2 - 2)\sin(n\pi) + 2\pi n\cos(n\pi)}{\pi n^3} - \frac{(n^2(-\pi^2) - 2)\sin(n(-\pi)) + 2(-\pi)n\cos(n(-\pi)}{\pi n^3}$$

$$= \frac{0 + 2\pi n(-1)^n}{n^3} - \frac{0 + 2(-\pi)n(-1)^n}{\pi n^3} = \frac{4\pi n(-1)^n}{\pi n^3} = \frac{4(-1)^n}{n^2} \quad .$$

So the Fourier series of x^2 is

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx \quad .$$

Finally to get the Fourier series of $2x^2 + 1$ we multiply by 2 and add 1:

$$2x^2 + 1 = 1 + \frac{2\pi^3}{3} + 8\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx \quad .$$

This is the **ans.**

Comments

1. An **even** more efficient way, since x^2 (and also $1 + 2x^2$) is an **even** function (note the pun), is to use

$$a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos nx \, dx \quad .$$

2. Many people gave the answer

$$\frac{2\pi^3}{3} + 8 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx \ + 1 \quad .$$

They got full credit, but stricly speaking, the +1 should be together with the other number $\frac{2\pi^3}{3}$. The pure number should stand in front.

3. Some people got the correct answer and then "improved" it by writing

$$1 + \frac{2\pi^3}{3} + 8 \sum_{k=0}^{\infty} \frac{(-1)^n}{(2k+1)^2} \cos(2k+1)x \quad .$$

This is **wrong**! (and they lost points for doing such nonsense. They blindly aped the simplification in one of the homework problem where there was $1 - (-1)^n$ at the top, but $(-1)^n$ is never zero, so it is completely a different situation!

If in doubt, do not "simplify" (and take a chance of producing nonsense that will cost you important points (or your job, in real life).