**Solutions to Dr. Z.'s Math 421(1), (Fall 2014, RU) REAL Quiz #4**

**1.** Show that the given functions are orthogonal on the given interval.

$$f_1(x) = e^{x^2}x^7 \quad , \quad f_2(x) = e^{-x^2}(x^4 + x^2 + 1) \quad , \quad [-2, 2] \quad .$$

**Sol.** We take the **inner-product**:

$$(f_1(x), f_2(x)) = \int_{-2}^{2} f_1(x)f_2(x)\,dx = \int_{-2}^{2} f_1(x)f_2(x)\,dx = \int_{-2}^{2} (e^{x^2}x^7)(e^{-x^2}(x^4 + x^2 + 1))\,dx$$

$$= \int_{-2}^{2} e^{x^2-x^2}x^7(x^4 + x^2 + 1)\,dx = \int_{-2}^{2} (x^{11} + x^9 + x^7)\,dx$$

$$= \frac{x^{12}}{12} + \frac{x^{10}}{10} + \frac{x^8}{8}\Big|_{-2}^{2} = \frac{2^{12} - (-2)^{12}}{12} + \frac{2^{10} - (-2)^{10}}{10} + \frac{2^8 - (-2)^8}{8} = 0 \quad .$$

Since the inner product is zero, the two functions are **orthogonal**.

**Comment 1**: A faster way, once you get to $\int_{-2}^{2}(x^{11} + x^9 + x^7)\,dx$ is to say that the **integrand** is an **odd function** and the interval of integration is symmetric (of the form $(-p, p)$, in this example, $p = 2$), so the integral must be zero.

**Comment 2**: Almost everyone got it right.

**2.** Show that $\{\cos nx\}$, $n = 1, 2, \ldots$ is orthogonal over the interval $[0, \pi]$. Also find the norm of each function.

**Sol.** We need to take two **different**, typical members of this family, so let's call them $\cos nx$ and $\cos mx$, where $n \neq m$. We have to show that $(\cos mx, \cos nx) = 0$.

$$(\cos mx, \cos nx) = \int_0^{\pi} \cos mx \cos nx\,dx \quad .$$

We now use the famous trig, identity, recently entered into the cheatsheet:

$$\cos A \cos B = \frac{1}{2}(\cos(A + B) + \cos(A - B)) \quad .$$

So

$$\cos mx \cos nx = \frac{1}{2}(\cos(m + n)x + \cos(m - n)x) \quad .$$

Going back to the integral:

$$\int_0^{\pi} \cos mx \cos nx\,dx = \frac{1}{2}\int_0^{\pi} (\cos(m + n)x + \cos(m - n)x) =$$

$$\frac{1}{2}\left(\frac{\sin(m + n)x}{m + n} + \frac{\sin(m - n)x}{m - n}\right)\Big|_0^{\pi} =$$

$$\frac{1}{2}\left(\frac{\sin(m+n)\pi}{m+n}+\frac{\sin(m-n)\pi}{m-n}\right)-\frac{1}{2}\left(\frac{\sin(m+n)\cdot 0}{m+n}+\frac{\sin(m-n)\cdot 0}{m-n}\right)=0-0=0 \quad ,$$

Since both $m+n$ and $m-n$ are integers, so both $\sin(m+n)\pi=0$ and $\sin(m-n)\pi=0$. Of course $\sin 0=0$.

Since this is true for **any** two different members of the family, the family is indeed ortogonal.

It still remains to find the norm, $||\cos nx||$. We firs find the **norm-squared**.

$$||\cos nx||^2=\int_0^\pi \cos^2 nx\,dx \quad .$$

Using the famous trig-identity:

$$\cos^2 A=\frac{1+\cos 2A}{2} \quad ,$$

we get

$$||\cos nx||^2=\int_0^\pi \frac{1+\cos 2nx}{2}\,dx=\frac{1}{2}\left(x+\frac{\sin 2nx}{2n}\right)\Big|_0^\pi=\frac{1}{2}\left(\pi-0+\frac{\sin 2n(\pi)-\sin 0}{2n}\right)=\frac{\pi}{2} \quad .$$

Taking square-root:

$$||\cos nx||=\sqrt{\frac{\pi}{2}}$$

**Ans. to 2**: The family is indeed an orthogonal family of functions, and the norm of $\cos nx$ is $\sqrt{\frac{\pi}{2}}$.

**Comment**: About %50 of the people got it completely. Most people got the norm, but quite a few people messed up with the inner-product of $\cos nx$ and $\cos mx$.