Solutions to Dr. Z.'s Math 421(2) REAL Quiz #1

1. Using the **definition** find the Laplace transform $\mathcal{L}\{f(t)\}$ (alias $F(s)$) of

$$f(t) = \begin{cases} -1, & \text{if } 0 \le t \le 2; \\ 1, & \text{if } t \ge 2. \end{cases}$$

Sol.:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt \;=\; \int_0^2 e^{-st}(-1) \;+\; \int_2^\infty e^{-st}(1)$$

$$= \frac{-e^{-st}}{-s}\Big|_0^2 + \frac{e^{-st}}{-s}\Big|_2^\infty$$

$$\frac{e^{-st}}{s}\Big|_0^2 - \frac{e^{-st}}{s}\Big|_2^\infty = \frac{e^{-s\cdot 2}}{s} - \frac{e^{-s\cdot 0}}{s} - \frac{e^{-s\cdot\infty}}{s} + \frac{e^{-s\cdot 0}}{s}$$

$$\frac{e^{-2s}}{s} - \frac{1}{s} - 0 + \frac{e^{-2s}}{s} = 2\frac{e^{-2s}}{s} - \frac{1}{s} = \frac{2e^{-2s} - 1}{s} \quad .$$

Ans. to 1:$F(s) = \frac{2e^{-2s}-1}{s}$.

2. Approximate, with mesh-size $h = 1$, the solution of the boundary-value problem

$$u_{xx} + u_{yy} = 0 \quad , \quad 0 < x < 2 \quad , \quad 0 < y < 2 \quad ;$$

subject to the boundary conditions

$$u(0, y) = 1 \quad , \quad 0 < y < 2 \quad ; \quad u(2, y) = 2 \quad , \quad 0 < y < 2 \quad ;$$

$$u(x, 0) = -1 \quad , \quad 0 < x < 2 \quad ; \quad u(x, 2) = 3 \quad , \quad 0 < x < 2 \quad .$$

Sol. to 2: The boundary points are $(1, 0), (2, 1), (1, 2), (0, 1)$. We have

$$u(1, 0) = -1 \quad , \quad u(2, 1) = 2 \quad , \quad u(1, 2) = 3 \quad , \quad u(0, 1) = 1 \quad ,$$

The only interior point is $(1, 1)$ and we have that $u(1, 1)$ is **approximately**

$$\frac{u(1, 0) + u(1, 2) + u(0, 1) + u(2, 1)}{4} = \frac{(-1) + 3 + 1 + 2}{4} = \frac{5}{4} \quad .$$

Ans to 2. $u(1, 1)$ is approximately $\frac{5}{4} = 1.25$.

Comment: Most people chose to do this easy version. For the original version (that I gave you the option to change to), $3u_{xx} + u_{yy} = 0$, you approximate the differential operator by

$$3\frac{u(x + h, y) - 2u(x, y) + u(x - h, y)}{h^2} + \frac{u(x, y + h) - 2u(x, y) + u(x, y - h)}{h^2}$$

$$= \frac{3u(x+h,y) + 3u(x-h,y) + u(x,y+h) + u(x,y-h) - 8u(x,y)}{h^2}$$

So, setting this equal to 0 (h^2 disappears) we get

$$u(x,y) \approx \frac{3u(x+h,y) + 3u(x-h,y) + u(x,y+h) + u(x,y-h)}{8}$$

So $u(1,1)$ is **approximately**

$$\frac{3u(2,1) + 3u(0,1) + u(1,2) + u(1,0)}{8} = \frac{3 \cdot 2 + 3 \cdot 1 + 3 + (-1)}{8} = \frac{6+3+3-1}{8} = \frac{11}{8} =$$

Ans. to the original version: $u(1,1)$ is approximately $\frac{11}{8}$.