Solutions to Dr. Z.'s Math 421(1) REAL Quiz #1

1. Using the **definition** find the Laplace transform $\mathcal{L}\{f(t)\}$ (alias $F(s)$) of $f(t) = e^{-t} + 3$.

Sol.

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}(e^{-t}+3)\,dt = \int_0^\infty e^{-st}e^{-t}\,dt + \int_0^\infty e^{-st}\cdot 3\,dt = \int_0^\infty e^{-st-t}\,dt + 3\int_0^\infty e^{-st}\,dt =$$

$$\int_0^\infty e^{-(s+1)t}\,dt \;+\; 3\int_0^\infty e^{-st}\,dt = \frac{e^{-(s+1)t}}{-(s+1)}\Big|_0^\infty + 3\frac{e^{-st}}{-s}\Big|_0^\infty$$

$$= \frac{e^{-\infty}}{-(s+1)} - \frac{e^0}{-(s+1)} + 3\frac{e^{-\infty}}{-s} - 3\frac{e^0}{-s} = \frac{1}{s+1} + \frac{3}{s} \quad .$$

Ans. to 1.: $\frac{1}{s+1} + \frac{3}{s}$.

2. Approximate, with mesh-size $h = 1$, the solution of the boundary-value problem

$$u_{xx} + u_{yy} = 0 \quad , \quad 0 < x < 2 \quad , \quad 0 < y < 2 \quad ;$$

subject to the boundary conditions

$$u(0,y) = 1 \quad , \quad 0 < y < 2 \quad ; \quad u(2,y) = 2 \quad , \quad 0 < y < 2 \quad ;$$

$$u(x,0) = -1 \quad , \quad 0 < x < 2 \quad ; \quad u(x,2) = 3 \quad , \quad 0 < x < 2 \quad .$$

Sol. to 2: The boundary points are $(1,0), (2,1), (1,2), (0,1)$. We have

$$u(1,0) = -1 \quad , \quad u(2,1) = 2 \quad , \quad u(1,2) = 3 \quad , \quad u(0,1) = 1 \quad ,$$

The only interior point is $(1,1)$ and we have that $u(1,1)$ is **approximately**

$$\frac{u(1,0)+u(1,2)+u(0,1)+u(2,1)}{4} = \frac{(-1)+3+1+2}{4} = \frac{5}{4} \quad .$$

Ans to 2. $u(1,1)$ is approximately $\frac{5}{4} = 1.25$.

Comment: Most people chose to do this easy version. For the original version (that I gave you the option to change to), $u_{xx} + 2u_{yy} = 0$, you approximate the differential operator by

$$\frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2} + 2\frac{u(x,y+h) - 2u(x,y) + u(x,y-h)}{h^2}$$

$$= \frac{u(x+h,y) + u(x-h,y) + 2u(x,y+h) + 2u(x,y-h) - 6u(x,y)}{h^2}$$

So, setting this equal to 0 (h^2 disappears) we get

$$u(x,y) \approx \frac{u(x+h,y) + u(x-h,y) + 2u(x,y+h) + 2u(x,y-h)}{6}$$

So $u(1,1)$ is **approximately**

$$\frac{2u(1,0) + 2u(1,2) + u(0,1) + u(2,1)}{6} = \frac{2\cdot(-1) + 2\cdot 3 + 1 + 2}{6} = \frac{7}{6} \quad .$$