**Solutions to Dr. Z.'s MATH 421 (2) Exam 2, Tue., Nov. 22, 2011, 12:00noon-1:20pm, SEC 216**

**1**. (15 pts.) Find the general expression, in polar coordinates, for the steady-state temperature $u(r,\theta)$ in a circular plate of radius 3, if the temperature on the circumference $r = 3$ is given by $u(3,\theta) = \cos\theta - 2\sin 3\theta$

---

**Ans.**: $u(r,\theta) = \frac{r}{3}\cos\theta - 2(\frac{r}{3})^3 \sin 3\theta$, or $u(r,\theta) = \frac{1}{3}r\cos\theta - \frac{2}{27}r^3\sin 3\theta$ .

---

**Sol.** This is a *piece of cake* using Dr. Z.'s shorcut method. The boundary condition $f(\theta) = \cos\theta - 2\sin 3\theta$ happens to be a finite linear combination of pure sine and cosine waves, so its Fourier Series equals itself!

So all we have to do to get $u(r,\theta)$ out of $f(\theta) = u(3,\theta)$ is stick "$(r/3)^n$" in front of any term of the form $\sin n\theta$, or $\cos n\theta$ , that's it.

In this problem there are only two terms in $f(\theta)$, both of them pure waves (one cosine, the other sine):

For the term $\cos\theta$, $n = 1$ so we multiply it by $(r/3)^1 = \frac{r}{3}$.

For the term $-2\sin 3\theta$, $n = 3$ so we multiply it by $(r/3)^3 = \frac{r^3}{27}$.

that's is!

**Comment**. **1.** Be careful before you use the shortcut. If it would have been $u(3,\theta) = \cos(\theta/2)$ or $u(3,\theta) = \cos(\pi\theta)$, then you have to it the long way.

**2.** If some of $f(\theta)$ is pure sine and cosine waves and the rest is not, you can make life much easier, by applying the shortcut to these terms, and applying the long way only to the rest, and at the end adding things up. This is legitimate because of the superpositionprinciple.

**3.** Reminding you that if $f(\theta)$ is **not** a finite linear combination of pure sine and/or cosine waves, then you would first need to find the full Fourier series, using the stuff from lecture 8, getting

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta \quad ,$$

where $a_n$ and $b_n$ are some *symbolic* (in $n$) expression in $n$ that you would get from the integrations given in the cheatsheet. Having done that, to get $u(r,\theta)$ you stick $(r/c)^n$ after the $a_n$ and $b_n$ above getting

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n (\frac{r}{c})^n \cos n\theta + \sum_{n=1}^{\infty} b_n (\frac{r}{c})^n \sin n\theta \quad .$$

**2.** (15 points altogether) (a) (8 points) Show that $\{\sin nx\}$, $n = 1, 2, \ldots$ is orthogonal over the interval $[0, \pi]$.

(b) Find the norm of each function.

---

**Ans. to (b)**: $||\sin nx|| = \sqrt{\frac{\pi}{2}}$

---

**Sol. of a.**:

$$(\sin nx, \sin mx) = \int_0^\pi (\sin nx)(\sin mx)\, dx \quad .$$

From the cheatsheet (or from your head):

$$(\sin A)(\sin B) = \frac{1}{2}(\cos(A-B) - \cos(A-B)) \quad .$$

So

$$(\sin nx)(\sin mv) = \frac{1}{2}(\cos(n-m)x - \cos(n+m)x) \quad .$$

So (of course $n \neq m$, so we can divide by $n-m$)

$$(\sin nx, \sin mx) = \int_0^\pi (\sin nx)(\sin mx)\, dx = \frac{1}{2}\int_0^\pi (\cos(n-m)x - \cos(n+m)x)\, dx = \frac{1}{2}\left(\frac{\sin(n-m)x}{n-m} - \frac{\sin(n+m)x}{n+m}\right)\Big|_0^\pi$$

$$= \frac{1}{2}\left(\frac{\sin(n-m)\pi - \sin 0}{n-m} - \frac{\sin(n+m)\pi - \sin 0}{n+m}\right) = 0 \quad .$$

since $m$ and $n$ are both integers, hence so are $m-n$ and $m+n$ so $\sin(n-m)\pi = 0$ and $\sin(n+m)\pi = 0$.

**Sol. of b.**:

$$||\sin nx||^2 = (\sin nx, \sin nx) = \int_0^\pi \sin^2 nx\, dx \quad .$$

From the cheatsheet, or from your head

$$\sin^2 A = \frac{1 - \cos 2A}{2} \quad ,$$

so

$$\sin^2 nx = \frac{1 - \cos 2nx}{2} \quad .$$

Going back to the integral:

$$||\sin nx||^2 = (\sin nx, \sin nx) = \int_0^\pi \sin^2 nx = \frac{1}{2}\int_0^\pi (1 - \cos 2nx)\, dx = \frac{1}{2}\left(x - \frac{\sin 2nx}{2n}\right)\Big|_0^\pi = \frac{1}{2}(\pi - 0 - (0-0)) = \frac{\pi}{2} \quad .$$

Taking the square-root, we get

$$||\sin nx|| = \sqrt{\frac{\pi}{2}} \quad .$$

**Comment**: Some people forgot to take the square-root at the end.

**3.** (15 points) Find the **complex** Fourier series of the function $f(x) = x$ on the interval $-\pi < x < \pi$.

---

**Ans.**:

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{i(-1)^n}{n} e^{inx} \quad .$$

---

**Sol.** Recall (from your head or from the formula sheet) that the **complex Fourier series** of a function defined in $-\pi < x < \pi$ is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad ,$$

where the coefficients $c_n$ are computed by the formula

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx \quad .$$

In this problem, $f(x) = x$, so

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x e^{-inx}\, dx \quad .$$

We must use the formula from the formula sheet (in real life, we would use Maple or Mathematica or Sage)

$$\int x e^{cx}\, dx = \frac{(-1+cx)\, e^{cx}}{c^2} + C = x\frac{e^{cx}}{c} - \frac{e^{cx}}{c^2} + C \quad .$$

(Of course, this is only valid for $c \neq 0$).

Here $c = -in$, so

$$\int_{-\pi}^{\pi} x e^{-inx}\, dx = \left( x\frac{e^{-inx}}{-in} - \frac{e^{-inx}}{(-in)^2} \right) \Big|_{-\pi}^{\pi} \quad .$$

This equals

$$\left( (\pi)\frac{e^{-in\pi}}{-in} - \frac{e^{-in\pi}}{(-in)^2} \right) - \left( (-\pi)\frac{e^{-in(-\pi)}}{-in} - \frac{e^{-in(-\pi)}}{(-in)^2} \right)$$

$$= \left( (\pi)\frac{e^{-in\pi}}{-in} - \frac{e^{-in\pi}}{-n^2} \right) - \left( (-\pi)\frac{e^{-in(-\pi)}}{-in} - \frac{e^{in\pi)}}{-n^2} \right) \quad .$$

But $e^{in\pi} = e^{-in\pi}$ (they both equal $(-1)^n$) so the two second terms cancel, and we are left with

$$(\pi)\frac{e^{-in\pi}}{-in} - (-\pi)\frac{e^{-in(-\pi)}}{-in} = (\pi)\frac{(-1)^n}{-in} - (-\pi)\frac{(-1)^n}{-in} = (2\pi)\frac{(-1)^n}{-in} = (2\pi)\frac{(-1)^n i}{n} \quad .$$

Going back to $c_n$, we have

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}\,dx = \frac{1}{2\pi}(2\pi)\frac{(-1)^n i}{n} = \frac{(-1)^n i}{n} \quad .$$

Of course this is only valid if $n \neq 0$. The case $n = 0$ has to be done separately.

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = \frac{1}{2\pi}\frac{x^2}{2}\Big|_{-\pi}^{\pi} = \frac{1}{2\pi}\frac{\pi^2 - (-\pi)^2}{2} = 0 \quad .$$

So the answer is

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{i(-1)^n}{n} e^{inx} \quad .$$

**Comment**: Many people who forgot to put $n \neq 0$. This time I didn't take any points off, but in the Final exam, I will! When $n = 0$ , $\frac{i(-1)^n}{n}$ is nonsense!

**4.** (15 points) The **Hermite** polynomials $H_n(x)$ are defined by

$$H_n(x) = 2xH_{n-1}(x) - 2(n-1)H_{n-2}(x)$$

with initial conditions $H_0(x) = 1, H_1(x) = 2x$. Find $H_4(x)$.

---

**Ans.**: $H_4(x) = 16x^4 - 48x^2 + 12$

---

**Sol.** We first must find $H_2(x)$. Using the **recurrence** with $n = 2$, we get

$$H_2(x) = 2xH_{2-1}(x) - 2(2-1)H_{2-2}(x) = 2xH_1(x) - 2H_0(x) = 2x(2x) - 2(1) = 4x^2 - 2 \quad .$$

Next we must find $H_3(x)$. Using the **recurrence** with $n = 3$, we get

$$H_3(x) = 2xH_{3-1}(x) - 2(3-1)H_{3-2}(x) = 2xH_2(x) - 4H_1(x) = 2x(4x^2-2) - 4(2x) = 8x^3 - 4x - 8x = 8x^3 - 12x \quad .$$

Finally, using $H_2(x)$ and $H_3(x)$ that we already know, we compute $H_4(x)$:

$$H_4(x) = 2xH_{4-1}(x) - 2(4-1)H_{4-2}(x) = 2xH_3(x) - 6H_2(x) =$$

$$2x(8x^3 - 12x) - 6(4x^2 - 2) = 16x^4 - 24x^2 - 24x^2 + 12 = 16x^4 - 48x^2 + 12 \quad .$$

This is the final answer.

**Comment**: Some people kept the $n$, having it show up in the answer. $H_4(x)$ is a *specific* polynomial of $x$ with *numerical* coefficients. At each use of the recurrence, $n$ has to be replaced by the appropriate specific integer, as in the solution above.

**5.** (15 points) Solve the boundary value pde problem:

$$4u_{xx} = u_{tt} \quad , 0 < x < \pi \quad , \quad t > 0 \quad ;$$

$$u(0,t) = 0 \quad , \quad u(\pi,t) = 0 \quad , \quad t > 0 \quad ;$$

$$u(x,0) = \sin 3x + \sin 4x \quad , \quad u_t(x,0) = \sin 2x - \sin 5x \quad , \quad 0 < x < \pi \quad .$$

---

**Ans**: $u(x,t) = \sin 3x \cos 6t + \sin 4x \cos 8t + \frac{1}{4} \sin 2x \sin 4t - \frac{1}{10} \sin 5x \sin 10t$

---

**Sol.** This is the wave equation with $a = 2$.

It is a *piece of cake* if you use Dr. Z.'s amazing shortcut. Since both $u(x,0)$ (that happens to be $\sin 3x + \sin 4x$ in this problem), and $u_t(x,0)$ (that happens to be $\sin 2x - \sin 5x$ in this problem) are already in Fourier-Sine expansion, leave them alone, and to get $u(x,t)$ change each $\sin nx$ in the $u(x,0) = f(x)$ part into $\sin nx \cos(nat)$ and change each $\sin nx$ in $g(x)$ featuring in the initial condition $u_t(x,0) = g(x)$ into $\sin nx \sin(nat)/(na)$.

In $f(x)$:

For the $\sin 3x$ piece, $n = 3$ so we multiply it by $\cos(3 \cdot 2t) = \cos(6t)$, getting $\sin 3x \cos 6t$

For the $\sin 4x$ piece, $n = 4$ so we multiply it by $\cos(4 \cdot 2 \cdot t) = \cos(8t)$, getting $\sin 4x \cos 8t$ .

In $g(x)$:

For the $\sin 2x$ piece, $n = 2$ so we multiply it by $\sin(2 \cdot 2t)/4 = \sin(4t)/4$, getting $\frac{1}{4} \sin 2x \sin 4t$

For the $-\sin 5x$ piece, $n = 5$ so we multiply it by $\sin(2 \cdot 5t)/10 = \sin(10t)/10$, getting $-\frac{1}{10} \sin 5x \sin 10t$.

Adding everything up, yields the answer:

$$u(x,t) = \sin 3x \cos 6t + \sin 4x \cos 8t + \frac{1}{4} \sin 2x \sin 4t - \frac{1}{10} \sin 5x \sin 10t \quad .$$

**6.** (15 points) Find the Fourier series of $f(x) = e^{2x} + 5$ over the interval $(-\pi, \pi)$.

**Hint:** You may use the ready-made formulas (that in real life Maple can find)

$$\int_{-\pi}^{\pi} e^{2x} \cos nx = \frac{2(-1)^n (e^{2\pi} - e^{-\pi})}{n^2 + 4}$$

$$\int_{-\pi}^{\pi} e^{2x} \sin nx = \frac{-n(-1)^n (e^{2\pi} - e^{-\pi})}{n^2 + 4}$$

---

**Ans.**:

$$5 + \frac{e^{2\pi} - e^{-2\pi}}{4\pi} + \frac{2(e^{2\pi} - e^{-2\pi})}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + 4} \cos nx \quad - \quad \frac{(e^{2\pi} - e^{-2\pi})}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2 + 4} \sin nx \quad .$$

---

**Sol.** We use the formula for the full Fourier Series of a function over the interval $(-\pi, \pi)$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad ,$$

where the coefficients are given by:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx \quad ,$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx \quad ,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx \quad .$$

Before we start, let's see whether we can use the shortcut. Of course not, since $e^{2x}$ is not a pure sine or pure cosine wave. **But** 5 being a number is! So, while it is not mandatory, it is more efficient to first find the Fourier Series of $e^{2x}$ and "at the end of the day", add 5 to it.

First:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{2x}\, dx = (\frac{1}{\pi}) \frac{e^{2x}}{2} \Big|_{-\pi}^{\pi} = \frac{e^{2\pi} - e^{-2\pi}}{2\pi} \quad .$$

Next (using the supplied formula)

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\frac{2(-1)^n(e^{2\pi}-e^{-\pi})}{n^2+4} = \frac{2(-1)^n(e^{2\pi}-e^{-\pi})}{\pi(n^2+4)} \quad .$$

$$b_n = \frac{1}{\pi}\cdot\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\cdot\frac{-(-1)^n n(e^{2\pi}-e^{-\pi})}{n^2+4} = -\frac{(-1)^n n(e^{2\pi}-e^{-\pi})}{\pi(n^2+4)} \quad .$$

Combining, and taking all the stuff that **does not** involve $n$ out of the $\sum$, we get:

$$e^{2x} = \frac{e^{2\pi}-e^{-2\pi}}{4\pi} + \frac{2(e^{2\pi}-e^{-2\pi})}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+4}\cos nx - \frac{(e^{2\pi}-e^{-2\pi})}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^2+4}\sin nx \quad .$$

**Finally**, adding 5 gives:

$$e^{2x}+5 = 5+\frac{e^{2\pi}-e^{-2\pi}}{4\pi}+\frac{2(e^{2\pi}-e^{-2\pi})}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+4}\cos nx \quad - \quad \frac{(e^{2\pi}-e^{-2\pi})}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^2+4}\sin nx \quad .$$

**7.** (10 points) Set up the boundary value problem for the steady-state temperature $u(x, y)$, where a thin rectangular plate coincides with the region in the $xy$-plane defined by $0 \leq x \leq 30, 0 \leq y \leq 40$ if The bottom, left and right sides are insulated, while the top side is held at temperature $e^x$.

---

**Ans.**: pde: $u_{xx} + u_{yy} = 0$, $0 < x < 30, 0 < y < 40$.

boundary conditions: $u_y(x, 0) = 0 \quad (0 < x < 30)$, $u_x(0, y) = 0 \quad (0 < y < 40) \quad , \quad u_x(30, y) = 0 \quad (0 < y < 40)$, $u(x, 40) = e^x \quad (0 < x < 30)$.

---

**Sol.**: From the physics of heat, we know that *insulating* means that the transversal derivative along the side is 0.

**Comment**: It is very important to do "transversal", i.e. the derivative with respect to the *other* variable. So people who wrote, for example, $u_x(x, 0) = 0 \quad (0 < x < 30)$ were wrong!