Solutions to Dr. Z.'s Math 421(2), Exam #1

1. (15 points) Use any method to compute $\mathcal{L}\{f(t)\}$ if

$$f(t) = (e^t + 1)(e^{2t} + 1) + t^3(t + 1) \quad .$$

Sol. First use algebra to **expand**:

$$f(t) = e^{3t} + e^t + e^{2t} + 1 + t^4 + t^3 \quad .$$

Now use the **table** to get

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{3t} + e^t + e^{2t} + 1 + t^4 + t^3\} = \mathcal{L}\{e^{3t}\} + \mathcal{L}\{e^t\} + \mathcal{L}\{e^{2t}\} + \mathcal{L}\{1\} + \mathcal{L}\{t^4\} + \mathcal{L}\{t^3\}$$

$$= \frac{1}{s-3} + \frac{1}{s-1} + \frac{1}{s-2} + \frac{1}{s} + \frac{24}{s^5} + \frac{6}{s^4} \quad .$$

Ans. to 1.: $\frac{1}{s-3} + \frac{1}{s-1} + \frac{1}{s-2} + \frac{1}{s} + \frac{24}{s^5} + \frac{6}{s^4}$.

2. (15 points) Solve the initial value problem:

$$y'' - y = 0 \quad , \quad y(0) = 2 \quad , \quad y'(0) = 0 \quad .$$

Sol.: Applying $\mathcal{L}$:

$$\mathcal{L}\{y''\} - \mathcal{L}\{y\} = \mathcal{L}\{0\} \quad .$$

Putting, as usual, $\mathcal{L}\{y\} = Y$, and using the formula for $\mathcal{L}\{y''\}$:

$$s^2Y - sy(0) - y'(0) - Y = 0 \quad .$$

So

$$s^2Y - 2s - Y = 0 \quad .$$

Moving $2s$ to the right:

$$(s^2 - 1)Y = 2s \quad .$$

Solving for Y:

$$Y = \frac{2s}{s^2 - 1} \quad .$$

Factorizing:

$$Y = \frac{2s}{(s-1)(s+1)} \quad .$$

Partial fractions:

$$\frac{2s}{(s-1)(s+1)} = \frac{A}{s-1} + \frac{B}{s+1} \quad .$$

Common denominator:

$$\frac{2s}{(s-1)(s+1)} = \frac{A(s+1) + B(s-1)}{(s-1)(s+1)} \quad .$$

Equating the tops:

$$2s = A(s+1) + B(s-1) \quad .$$

When $s = 1$: $2 = A(1+1)$, so $A = 1$. When $s = -1$: $-2 = B(-1-1)$, so $B = 1$. Going back to the template

$$Y = \frac{1}{s-1} + \frac{1}{s+1} \quad .$$

Applying $\mathcal{L}^{-1}$ we finally get (thanks to the formula $\mathcal{L}^{-1}\{\frac{1}{s-a}\} = e^{at}$):

$$y(t) = \mathcal{L}^{-1}\{\frac{1}{s-1} + \frac{1}{s+1}\} = e^t + e^{-t} \quad .$$

Ans. to 2.: $y(t) = e^t + e^{-t}$.

3a. (13 points) Solve the initial value problem

$$y' + y = f(t) \quad , \quad y(0) = 0 \quad , where \quad f(t) = \begin{cases} 1, & \text{if } 0 \le t \le 1; \\ -1, & \text{if } 1 \le t < \infty. \end{cases} \quad .$$

Sol.: $f(t) = (1 - \mathcal{U}(t-1)) + (-1)(\mathcal{U}(t-1) - \mathcal{U}(t-\infty)) = 1 - 2\mathcal{U}(t-1)$. So $\mathcal{L}\{f(t)\} = \frac{1-2e^{-s}}{s}$. Apply $\mathcal{L}$ to the DE

$$\mathcal{L}\{y' + y\} = \mathcal{L}\{f(t)\} \quad .$$

Putting, as usual, $\mathcal{L}\{y\} = Y$,

$$sY - y(0) + Y = \frac{1 - 2e^{-s}}{s} \quad ,$$

$$(s+1)Y = \frac{1 - 2e^{-s}}{s} \quad .$$

Solving for Y:

$$Y = \frac{1 - 2e^{-s}}{s(s+1)} = \frac{1}{s(s+1)} - 2\frac{e^{-s}}{s(s+1)} \quad .$$

Let's call

$$F(s) = \frac{1}{s(s+1)} \quad .$$

By partial fractions, $F(s) = \frac{1}{s} - \frac{1}{s+1}$ (you do it!). So $f(t) = 1 - e^{-t}$. Now

$$y(t) = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\{\frac{1}{s(s+1)} - 2\frac{e^{-s}}{s(s+1)}\} = \mathcal{L}^{-1}\{\frac{1}{s(s+1)}\} - 2\mathcal{L}^{-1}\{\frac{e^{-s}}{s(s+1)}\} \quad .$$

We already know that $\mathcal{L}^{-1}\{\frac{1}{s(s+1)}\} = 1 - e^{-t}$. As for $\mathcal{L}^{-1}\{\frac{e^{-s}}{s(s+1)}\}$ we use $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a)$, so

$$\mathcal{L}^{-1}\{\frac{e^{-s}}{s(s+1)}\} = f(t-1)\mathcal{U}(t-1) \quad .$$

So we get:

$$\mathcal{L}^{-1}\{\frac{e^{-s}}{s(s+1)}\} = (1 - e^{-(t-1)})\mathcal{U}(t-1) \quad .$$

Combining:

$$y(t) = (1 - e^{-t}) - 2(1 - e^{-(t-1)})\mathcal{U}(t-1) \quad .$$

3b (2 points) What is $y(2)$?

$$y(2) = (1 - e^{-2}) - 2(1 - e^{-(2-1)})\mathcal{U}(1) \quad .$$

Of course $\mathcal{U}(1) = 1$, (since $1 > 0$) so:

$$y(2) = (1 - e^{-2}) - 2(1 - e^{-(2-1)}) = (1 - e^{-2}) - 2(1 - e^{-1}) = 1 - e^{-2} - 2 + 2e^{-1}) = -1 + \frac{2}{e} - \frac{1}{e^2} \quad .$$

Ans. to 3b: $y(2) = -1 + \frac{2}{e} - \frac{1}{e^2}$.

4. (15 points) Solve the following integral equation

$$f(t) = te^t + \int_0^t \tau f(t-\tau)\, d\tau \quad .$$

Sol. In the language of **convolution**:

$$f(t) = te^t + t * f(t) \quad .$$

Applying $\mathcal{L}$:

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{te^t\} + \mathcal{L}\{t * f(t)\} \quad .$$

Hence (using $\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$)

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{te^t\} + \mathcal{L}\{t\}\mathcal{L}\{f(t)\} \quad .$$

Putting $F(s) = \mathcal{L}\{f(t)\}$, we get

$$F(s) = \frac{1}{(s-1)^2} + \frac{1}{s^2}F(s) \quad .$$

Moving all the $F(s)$'s to the left side:

$$F(s) - \frac{1}{s^2}F(s) = \frac{1}{(s-1)^2} \quad .$$

Facoring out $F(s)$

$$\left(1 - \frac{1}{s^2}\right) F(s) = \frac{1}{(s-1)^2} \quad .$$

Simplifying:

$$\frac{s^2 - 1}{s^2}F(s) = \frac{1}{(s-1)^2} \quad .$$

Solving for $F(s)$:

$$F(s) = \frac{s^2}{(s^2-1)(s-1)^2} \quad .$$

Completely factorizing the bottom:

$$F(s) = \frac{s^2}{(s-1)(s+1)(s-1)^2} = \frac{s^2}{(s-1)^3(s+1)} \quad .$$

For Partial fractions, we use the **template**:

$$\frac{s^2}{(s+1)(s-1)^3} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3} + \frac{D}{s+1} \quad .$$

Common denominator:

$$\frac{s^2}{(s+1)(s-1)^3} = \frac{A(s-1)^2(s+1) + B(s-1)(s+1) + C(s+1) + D(s-1)^3}{(s+1)(s-1)^3} \quad .$$

Equating tops:

$$s^2 = A(s-1)^2(s+1) + B(s-1)(s+1) + C(s+1) + D(s-1)^3 \quad .$$

When $s = 1$: $1 = C(2)$, so $C = \frac{1}{2}$. When $s = -1$, $1 = D(-8)$ so $D = -\frac{1}{8}$. The coeff. of s^3 in the left is 0 and in the right is $A+D$ so $A+D = 0$ and so $A = \frac{1}{8}$. When $s = 0$ we get $0 = A-B+C-D$, so $B = A + C - D = \frac{1}{8} + \frac{1}{2} + \frac{1}{8} = \frac{3}{4}$.

Going back to the template:

$$\frac{s^2}{(s+1)(s-1)^3} = \frac{1/8}{s-1} + \frac{3/4}{(s-1)^2} + \frac{1/2}{(s-1)^3} + \frac{-1/8}{s+1} \quad .$$

Now apply $\mathcal{L}^{-1}$ to get

$$f(t) = \mathcal{L}^{-1}\{\frac{1/8}{s-1} + \frac{3/4}{(s-1)^2} + \frac{1/2}{(s-1)^3} + \frac{-1/8}{s+1}\} = \frac{1}{8}e^t + \frac{3}{4}te^t + \frac{1}{4}t^2e^t - \frac{1}{8}e^{-t} \quad .$$

Ans. to 4.: $\frac{1}{8}e^t + \frac{3}{4}te^t + \frac{1}{4}t^2e^t - \frac{1}{8}e^{-t}$.

5. (15 points) Solve the initial value problem

$$y'' + 6y' + 10y = \delta(t-\pi) \quad , \quad y(0) = 0 \quad , \quad y'(0) = 0 \quad .$$

Sol.. Let $\mathcal{L}\{y(t)\} = Y(s)$. Applying $\mathcal{L}$:

$$\mathcal{L}\{y''\} + 6\mathcal{L}\{y'\} + 10\mathcal{L}\{y\} = e^{-\pi s} \quad .$$

$$s^2Y - sy(0) - y'(0) + 6(sY - y(0)) + 10Y = e^{-\pi s} \quad .$$

$$s^2Y + 6sY + 10Y = e^{-\pi s} \quad .$$

$$(s^2+6s+10)Y = e^{-\pi s} \quad .$$

Solving for Y:

$$Y = \frac{e^{-\pi s}}{s^2+6s+10} \quad .$$

Let's call

$$F(s) = \frac{1}{s^2+6s+10} \quad .$$

Complete the square: $s^2+6s+10 = (s+3)^2+1^2$, so

$$F(s) = \frac{1}{(s+3)^2+1} \quad .$$

From the tables $f(t) = e^{-3t}\sin t$, so

$$y(t) = \mathcal{L}^{-1}\{\frac{e^{-\pi s}}{s^2+6s+10}\} = f(t-\pi)\mathcal{U}(t-\pi) = e^{-3(t-\pi)}\sin(t-\pi)\mathcal{U}(t-\pi) \quad .$$

Finally, using the trig. identity: $\sin(t-\pi) = -\sin t$, we get

$$y(t) = -e^{-3(t-\pi)}\sin t\,\mathcal{U}(t-\pi) \quad .$$

Ans. to 5.: $y(t) = -e^{-3(t-\pi)}\sin t\,\mathcal{U}(t-\pi)$.

6. (15 points) Solve the following system of differential equations with the indicated initial values.

$$\frac{dx}{dt} = -x+5y \quad ,$$

$$\frac{dy}{dt} = -2x+y \quad ,$$

$$x(0) = 4 \quad , \quad y(0) = -2 \quad .$$

Sol.: Apply $\mathcal{L}$:

$$\mathcal{L}\{\frac{dx}{dt}\} = \mathcal{L}\{-x+5y\} \quad ,$$

$$\mathcal{L}\{\frac{dy}{dt}\} = \mathcal{L}\{-2x+y\} \quad .$$

As usual, define $\mathcal{L}\{x\} = X$, $\mathcal{L}\{y\} = Y$. Since $\mathcal{L}\{x'\} = sX - x(0)$, $\mathcal{L}\{y'\} = sY - y(0)$, we get

$$sX - 4 = -X + 5Y$$

$$sY + 2 = -2X + Y$$

Cleaning-up, moving all terms that do not involve X or Y to the right, we get the system

$$(i) \quad (s+1)X - 5Y = 4 \quad ,$$

$$(ii) \quad 2X + (s-1)Y = -2 \quad .$$

Let's use **elimination**. Multiplying Eq. (i) by 2 and Eq. (ii) by $s+1$ and **subtracting** we get

$$2(i): \qquad 2(s+1)X - 10Y = 8 \quad .$$

$$(s+1)(ii): \quad (s+1)2X + (s+1)(s-1)Y = -2(s+1) \quad .$$

$$2(i) - (s+1)(ii): \qquad -10Y - (s^2-1)Y = 8 + 2s + 2$$

Cleaning up:

$$-(s^2+9)Y = 2s + 10$$

Dividing both sides by $-(s^2+9)$ we get

$$Y = -\frac{2s+10}{s^2+9} \quad .$$

We still have to find X. Multiplying Eq. (i) by $(s-1)$ and Eq. (ii) by 5 and **adding** we get:

$$(s-1)(i): \qquad (s-1)(s+1)X - (s-1)5Y = (s-1)4 \quad ,$$

$$(5)(ii): \qquad 10X + 5(s-1)Y = -10 \quad .$$

$$(s-1)(i) + 5(ii): \qquad (s^2 - 1 + 10)X = 4s - 4 - 10$$

Cleaning up:

$$(s^2+9)X = 4s - 14$$

Solving for X:

$$X = \frac{4s-14}{s^2+9} \quad .$$

Now it is time to apply $\mathcal{L}^{-1}$.

$$x(t) = \mathcal{L}^{-1}\{\frac{4s-14}{s^2+9}\} = 4\mathcal{L}^{-1}\{\frac{s}{s^2+9}\} - 14\mathcal{L}^{-1}\{\frac{1}{s^2+9}\} = 4\cos 3t - 14\frac{\sin 3t}{3} = 4\cos 3t - \frac{14}{3}\sin 3t \quad .$$

$$y(t) = \mathcal{L}^{-1}\{\frac{-2s-10}{s^2+9}\} = -2\mathcal{L}^{-1}\{\frac{s}{s^2+9}\} - 10\mathcal{L}^{-1}\{\frac{1}{s^2+9}\} = -2\cos 3t - 10\frac{\sin 3t}{3} = -2\cos 3t - \frac{10}{3}\sin 3t \quad .$$

Ans. to 6: $x(t) = 4\cos 3t - \frac{14}{3}\sin 3t$, $y(t) = -2\cos 3t - \frac{10}{3}\sin 3t$.

7. (10 points) Find all the eigenvalues of the matrix

$$\begin{bmatrix} 7 & -4 \\ 12 & -7 \end{bmatrix} \quad ,$$

and determine a basis for each eigenspace.

Sol.: The **Characteristic matrix** is

$$\begin{bmatrix} 7-\lambda & -4 \\ 12 & -7-\lambda \end{bmatrix} \quad .$$

Taking its **determinant**, we have

$$\det \begin{bmatrix} 7-\lambda & -4 \\ 12 & -7-\lambda \end{bmatrix} = (7-\lambda)(-7-\lambda)-(-4)(12) = (\lambda-7)(\lambda+7)+48 = \lambda^2-49+48 = \lambda^2-1 \quad .$$

So the **characteristic equation** is:

$$\lambda^2 - 1 = 0 \quad .$$

Factorizing we get

$$(\lambda+1)(\lambda-1) = 0 \quad .$$

So the **eigenvalues** are $\lambda = -1$ and $\lambda = 1$.

For each of them we must still find a basis for the **eigenspace**.

For $\lambda = -1$, we have to find vector(s) $\begin{bmatrix} a \\ b \end{bmatrix}$, such that

$$\begin{bmatrix} 7 & -4 \\ 12 & -7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = (-1) \begin{bmatrix} a \\ b \end{bmatrix} \quad .$$

In everyday notation:

$$7a - 4b = -a \quad , \quad 12a - 7b = -b \quad .$$

Rearranging:

$$8a - 4b = 0 \quad , \quad 12a - 6b = 0 \quad .$$

The second equation is a multiple of the first one, so we can ignore it, and from the first equation, we get $b = 2a$. To make it nice, we can take a to be any non-zero number. Taking $a = 1$ gives $b = 2$, so a basis vector is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

For $\lambda = 1$, we have to find vector(s) $\begin{bmatrix} a \\ b \end{bmatrix}$, such that

$$\begin{bmatrix} 7 & -4 \\ 12 & -7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = (1) \begin{bmatrix} a \\ b \end{bmatrix} \quad .$$

In everyday notation:

$$7a - 4b = a \quad , \quad 12a - 7b = b \quad .$$

Rearranging:

$$6a - 4b = 0 \quad , \quad 12a - 8b = 0 \quad .$$

The second equation is a twice the first one, so we can ignore it, and from the first equation, we get $b = \frac{3}{2}a$. To make it nice, we can take a to be any non-zero number. Taking $a = 2$ gives $b = 3$, so a basis vector is $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Ans. to 7 The eigenvalues are $\lambda = -1$ and a basis for its eigenspace is $\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\}$, and $\lambda = 1$ and a basis for its eigenspace is $\{\begin{bmatrix} 2 \\ 3 \end{bmatrix}\}$.