Full Solutions to MATH 421(1), Fall 2014, Dr. Z.'s FINAL EXAM, Mon., Dec. 22, 2014, 8:00-11:00pm, SEC 212

1. (15 pts.) Find the general expression, in polar coordinates, for the steady-state temperature $u(r,\theta)$ in an **infinite** plate with a circular hole, centered at the origin, of radius 2, if the temperature on $r = 2$ is given by $u(2,\theta) = -7\cos\theta + \frac{1}{4}\sin 5\theta$.

Ans.: $u(r,\theta) = -\frac{14}{r}\cos\theta + \frac{8}{r^5}\sin(5\theta)$

$u(2,\theta) = -7\cos\theta + \frac{1}{4}\sin 5\theta$ is a linear combination of pure sine and cosine functions ($\sin n\theta$ and $\cos n\theta$ for **integer** n), so its Fourier Series is **itself!**

Since this is an **infinite** plate, we stick $(r/c)^{-n}$ in front of every $\cos n\theta$ or $\sin n\theta$, (here $c = 2$) so we get

$$u(r,\theta) = -7(\frac{r}{2})^{-1}\cos\theta + \frac{1}{4}(\frac{r}{2})^{-5}\sin 5\theta = -\frac{14}{r}\cos\theta + \frac{1}{4}(\frac{2}{r})^{5}\sin 5\theta$$
$$= -\frac{14}{r}\cos\theta + \frac{1}{4}\cdot\frac{2^5}{r^5}\sin 5\theta = -\frac{14}{r}\cos\theta + \frac{1}{4}\cdot\frac{32}{r^5}\sin 5\theta = -\frac{14}{r}\cos\theta + \frac{8}{r^5}\sin 5\theta$$

2. (15 points altogether) (a) (8 pts.) Show that the set of functions $\{\cos(n\pi x)\}$, $n = 1, 2, \ldots$ is orthogonal over the interval $[0, 1]$.

(b)(7 pts.) Find the norm of each function.

Ans. to (b): $||\cos(n\pi x)|| = \frac{\sqrt{2}}{2}$

Sol. to a: We must use the trig-identity:

$$\cos A \cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B)) \quad .$$

If $m \neq n$:

$$(\cos m\pi x, \cos n\pi x) = \int_0^1 \cos m\pi x \cos n\pi x =$$

$$\frac{1}{2}\int_0^1 (\cos(m\pi x + n\pi x) + \cos(m\pi x - n\pi x))\, dx = \frac{1}{2}\int_0^1 (\cos(m+n)\pi x + \cos(m-n)\pi x)\, dx$$

$$= \frac{1}{2}\frac{\sin(m+n)\pi x}{(m+n)\pi}\Big|_0^1 + \frac{1}{2}\frac{\sin(m-n)\pi x}{(m-n)\pi}\Big|_0^1$$

$$= \frac{1}{2}(\frac{\sin(m+n)\pi}{(m+n)\pi} - \frac{\sin 0}{(m+n)\pi} + \frac{1}{2}(\frac{\sin(m-n)\pi}{(m-n)\pi} - \frac{\sin 0}{(m-n)\pi} = 0 - 0 + 0 = 0 \quad ,$$

(since $m+n$ and $m-n$ are integers $\sin(m+n)\pi$ and $\sin(m-n)\pi$ are both 0).

Sol. to b: We must use the trig-identity:

$$\cos^2 A = \frac{1 + \cos 2A}{2} \quad .$$

$$||\cos n\pi x||^2 = (\cos n\pi x, \cos n\pi x) = \int_0^1 \cos^2 n\pi x = \frac{1}{2}\int_0^1 (1 + \cos 2n\pi x)\, dx$$

$$= \frac{1}{2}\left(x + \frac{\sin(2n\pi x)}{2n\pi}\right)\Big|_0^1 = \frac{1}{2}(1-0) + \frac{1}{2}(0-0) = \frac{1}{2} \quad .$$

Taking the square-root, we get

$$||\cos n\pi x|| = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2} \quad .$$

3. (15 points) Find the **complex** Fourier series of the function $f(x) = x+5$ on the interval $-1 < x < 1$.

(Warning: The interval in this problem is **not** $-\pi < x < \pi$).

Ans.:

$$5 + \frac{i}{\pi} \sum_{n=-\infty, n\neq 0}^{\infty} \frac{(-1)^n}{n} e^{in\pi x} \quad .$$

Sol.: Since the interval is **not** $(-\pi, \pi)$ but rather $(-1, 1)$ (so $L = 1$) we must fist cosider a brand-new function $g(x) = f(xL/\pi) = f(x/\pi)$. For future reference, $f(x) = g(\pi x)$.

So (replacing x by x/π in $f(x) = x + 5$:

$$g(x) = \frac{x}{\pi} + 5 \quad .$$

Since 5 is already in the correct form (it is its own complex-Fourier series), it is easier to forget about it and at the end add it back. So let's find the complex Fourier series

$$\frac{x}{\pi} = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad ,$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{x}{\pi} e^{-inx}\, dx \quad .$$

So

$$c_n = \frac{1}{2\pi^2} \int_{-\pi}^{\pi} x e^{-inx}\, dx \quad .$$

When $n = 0$ this is 0. When $n \neq 0$, from the formula sheet

$$\int x e^{cx}\, dx = \frac{1}{c} x e^{cx} - \frac{1}{c^2} e^{cx} \quad .$$

So (with $c = -in$):

$$\int_{-\pi}^{\pi} x e^{-inx}\, dx = \frac{1}{-in} x e^{-inx} \Big|_{-\pi}^{\pi} - \frac{1}{(-in)^2} e^{-inx} \Big|_{-\pi}^{\pi}$$

$$\frac{1}{-in}(\pi e^{-in\pi} - (-\pi e^{in\pi})) - \frac{1}{(-in)^2}(e^{-in\pi} - e^{in\pi}) =$$

$$\frac{1}{-in}(\pi(-1)^n + \pi(-1)^n)) - \frac{1}{(-in)^2}((-1)^n - (-1)^n) = \frac{1}{-in}(\pi(-1)^n + \pi(-1)^n)) + 0 = \frac{2i\pi(-1)^n}{n} \quad .$$

(Since $e^{i\pi} = -1$ and $e^{-i\pi} = -1$ so $e^{in\pi} = (-1)^n$ and $e^{-in\pi} = (-1)^n$, also $1/(-i) = i$.)
Going back to c_n we have

$$c_n = \frac{1}{2\pi^2}\frac{2i\pi(-1)^n}{n} = \frac{i}{\pi}\frac{(-1)^n}{n} \quad .$$

So in the nice interval $(-\pi, \pi)$:

$$\frac{x}{\pi} = \frac{i}{\pi}\sum_{n=-\infty, n\neq 0}^{\infty}\frac{(-1)^n}{n}e^{inx}$$

Putting back the 5, we have:

$$g(x) = 5 + \frac{i}{\pi}\sum_{n=-\infty, n\neq 0}^{\infty}\frac{(-1)^n}{n}e^{inx} \quad .$$

Finnaly, to get the original $f(x)$, we use $f(x) = g(n\pi x)$, and replace x by πx:

$$f(x) = 5 + \frac{i}{\pi}\sum_{n=-\infty, n\neq 0}^{\infty}\frac{(-1)^n}{n}e^{in\pi x} \quad .$$

This is the final **answer**.

4. (15 points) The **Kermit** polynomials $K_n(x)$ are defined by

$$K_n(x) = K_{n-1}(x) + K_{n-2}(x) + nxK_{n-3}(x)$$

with initial conditions $K_0(x) = 1, K_1(x) = 1, K_2(x) = 1$. Find $K_4(x)$.

Ans.: $K_4(x) = 7x + 3$

First we must find $K_3(x)$. Plugging-in $n = 3$ into the recurrence, we get:

$$K_3(x) = K_2(x) + K_1(x) + 3xK_0(x)$$

Using the available data:

$$K_3(x) = 1 + 1 + 3x(1) = 3x + 2 \quad .$$

Plugging-in $n = 4$ into the recurrence, we get:

$$K_4(x) = K_3(x) + K_2(x) + 4xK_1(x)$$

Using the already-known $K_3(x), K_2(x), K_1(x)$, we have:

$$K_4(x) = 3x + 2 + 1 + 4x(1) = 7x + 3 \quad .$$

5. (15 points) Solve the boundary value pde problem:

$$16u_{xx} = u_{tt} \quad , 0 < x < \pi \quad , \quad t > 0 \quad ;$$

$$u(0,t) = 0 \quad , \quad u(\pi,t) = 0 \quad , \quad t > 0 \quad ;$$

$$u(x,0) = \sin x + \sin 2x \quad , \quad u_t(x,0) = 20\sin 3x - 4\sin 4x \quad , \quad 0 < x < \pi \quad .$$

Ans:

$$u(x,t) = \sin x \cos 4t + \sin 2x \cos 8t + \frac{5}{3}\sin 3x \sin 12t - \frac{1}{4}\sin 4x \sin 16t$$

This is the **wave equation** with $a = 4$. Using Dr. Z.'s famous shortcut (since the boundary condtions are the usual ones from music) all we have to do is multiply each $\sin nx$ that shows up in $u(x,0)$ by $\cos(nat)$ and each $\sin nx$ that shows up in $u_t(x,0)$ by $\frac{\sin nat}{na}$.

For $u(x,0)$:

$\sin x$ becomes $\sin x \cos 4t$ (since now $n = 1$)

$\sin 2x$ becomes $\sin 2x \cos 8t$ (since now $n = 2$)

For $u_t(x,0)$:

$20\sin 3x$ becomes $20\sin 3x \frac{\sin 12t}{12}$ (since now $n = 3$). Simplifying this is $\frac{5}{3}\sin 3x \sin 12t$

$-4\sin 4x$ becomes $-4\sin 4x \frac{\sin 16t}{16}$ (since now $n = 4$). Simplifying this is $-\frac{1}{4}\sin 4x \sin(16t)$

By the **principle of superposition** we can combine them all and get the answer.

6. (15 points) Find the Fourier series of $f(x) = 10 + \cos x + 2\cos 3x + 6\sin x + 12\sin 3x$ over the interval $(-\pi, \pi)$.

Ans.: $10 + \cos x + 2\cos 3x + 6\sin x + 12\sin 3x$.

This is a **piece of cake**. Since the interval is the nice one, $(-\pi, \pi)$ and $f(x)$ is a linear combination of pure sine and pure cosine functions $\sin nx$ and $\cos nx$ for *integers* n, the **input equals the output**.

7. (15 points) Set up the boundary value problem for the steady-state temperature $u(x, y)$, where a thin rectangular plate coincides with the region in the xy-plane defined by $0 \leq x \leq 30, 0 \leq y \leq 40$ if the bottom and right sides are insulated, while the top side is held at temperature 200 and the left side is held at temperature 100.

Ans.: pde: $u_{xx} + u_{yy} = 0$

boundary conditions: $u_y(x, 0) = 0$ $(0 < x < 3)$, $u_x(30, y) = 0$ $(0 < y < 40)$, $u(x, 40) = 200$, $u(0, y) = 100$ $(0 < y < 10)$.

This was a pure physics problem. "Insulated means" $u_x(a, y) = 0$ for the side $x = a$ if it a vertical side $x = a$ and $u_y(x, b) = 0$ if it is a horizontal side.

8. (15 points) Use any method to compute $\mathcal{L}\{f(t)\}$ if

$$f(t) = (e^t + 1)(e^t - 1) + (t + 2)(t - 2) \quad .$$

Ans.: $\frac{1}{s-2} + \frac{2}{s^3} - \frac{5}{s}$.

We first use *algebra* to open-up all parentheses:

$$f(t) = (e^t)^2 - 1 + t^2 - 2^2 = e^{2t} - 1 + t^2 - 4 = e^{2t} + t^2 - 5 \quad .$$

We now use the **table**:

$$F(s) = \mathcal{L}\{e^{2t} + t^2 - 5\} = \mathcal{L}\{e^{2t}\} + \mathcal{L}\{t^2\} - 5\mathcal{L}\{1\} = \frac{1}{s-2} + \frac{2}{s^3} - \frac{5}{s} \quad .$$

9. (15 points) Use the **Laplace transform** (no credit for other methods!) to solve the pde

$$u_{xx} = u_{tt} \quad , 0 < x < \pi \quad , \quad t > 0 \quad ,$$

subject to the **boundary-conditions**

$$u_x(0,t) = 0 \quad , \quad u_x(\pi,t) = 0 \quad , \quad t > 0 \quad ,$$

and the **initial conditions**

$$u(x,0) = \cos x \quad , \quad u_t(x,0) = 0 \quad , \quad 0 < x < \pi \quad .$$

Ans.: $u(x,t) = \cos x \cos t$.

We first apply $\mathcal{L}$ with respect to the variable t, viewing x as a silent *parameter*. We put, as usual $\mathcal{L}\{u(x,t)\} = U(x,s)$.

$$\mathcal{L}\{u_{xx}\} = \mathcal{L}\{u_{tt}\} \quad .$$

This is:

$$U_{xx}(x,s) = s^2 U(x,s) - su(x,0) - u_t(x,0) \quad .$$

Using the *initial conditions* $u(x,0) = \cos x$, $u_t(x,0) = 0$ this becomes:

$$U_{xx}(x,s) = s^2 U(x,s) - s\cos x - 0 \quad .$$

So we have to solve the ode, in x (now s is a parameter)

$$U''(x) - s^2 U(x,s) = -s\cos x \quad .$$

This is an *inhomoegenous* ode. From the **boundary conditions** $u_x(0,t) = 0$, $u_x(\pi,t) = 0$, we get, by taking the Laplace transfrom, $U_x(0) = 0$, $U_x(\pi) = 0$.

The **general solution** of the homogeneous version is (the characteristic equation is $r^2 - s^2 = 0$ so $(r-s)(r+s) = 0$ giving $r = -s$, $r = s$)

$$c_1 e^{sx} + c_2 e^{-sx}$$

We next need a **Particular Solution**. Since the left side is a a multiple of $\cos x$, we use the **template**

$$U(x) = A\cos x \quad .$$

Plugging this into the ode gives

$$(A\cos x)'' - s^2(A\cos x) = -s\cos x \quad .$$

So:

$$-A\cos x - s^2 A\cos x = -s\cos x \quad .$$

Factoring out $\cos x$:

$$-A(1+s^2)\cos x = -s\cos x \quad .$$

Solving for A, we get

$$A = \frac{s}{s^2+1} \quad .$$

Going back to the template, a particular solution to the ode is:

$$U(x,s) = \frac{s}{s^2+1}\cos x \quad .$$

So the **General Solution** of the ode is:

$$U(x,s) = c_1 e^{sx} + c_2 e^{-sx} + \frac{s}{s^2+1}\cos x \quad .$$

We still need to find the constants c_1, c_2. First we find $U'(x,s)$:

$$U'(x,s) = c_1 s e^{sx} - c_2 s e^{-sx} - \frac{s}{s^2+1}\sin x \quad .$$

Pluging-in $x = 0$ and $x = \pi$ (since $\sin\pi = 0$) we get the system

$$s(c_1 - c_2) = 0 \quad , \quad s(c_1 e^{s\pi} - c_2 e^{-s\pi}) = 0 \quad .$$

Simplifying:

$$c_1 - c_2 = 0 \quad , \quad c_1 e^{s\pi} - c_2 e^{-s\pi} = 0 \quad ,$$

whose solution is $c_1 = 0$, $c_2 = 0$. Going back to the general solution, we have:

$$U(x,s) = (0)e^{sx} + (0)e^{-sx} + \frac{s}{s^2+1}\cos x = \frac{s}{s^2+1}\cos x \quad .$$

We are almost done. To get $u(x,t)$, we apply $\mathcal{L}^{-1}$:

$$u(x,t) = \mathcal{L}^{-1}\{\frac{s}{s^2+1}\} = \cos x\,\mathcal{L}^{-1}\{\frac{s}{s^2+1}\} \quad ,$$

since $\cos x$ is **just** a constant (from the point of view of s and t). From the table (in the formula sheet or your head): $\mathcal{L}^{-1}\{\frac{s}{s^2+1}\} = \cos t$, so

$$u(x,t) = \cos x\cos t \quad .$$

10. (15 points) Solve the following integral equation

$$f(t) = 4e^t + \int_0^t \tau f(t-\tau)\, d\tau \quad .$$

Ans.: $f(t) = e^{-t} + 3e^t + 2te^t$.

In **convolution notation** this is:

$$f(t) = 4e^t + t * f(t) \quad .$$

Applying $\mathcal{L}$ we get

$$\mathcal{L}\{f(t)\} = 4\mathcal{L}\{e^t\} + \mathcal{L}\{t * f(t)\} \quad .$$

Since $\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$. this becomes

$$\mathcal{L}\{f(t)\} = \frac{4}{s-1} + \mathcal{L}\{t\}\mathcal{L}\{f(t)\} \quad ,$$

So (as usual, let $\mathcal{L}\{f(t)\} = F(s)$),

$$F(s) = \frac{4}{s-1} + \frac{1}{s^2}F(s) \quad .$$

The next step is to *solve* for $F(s)$:

$$F(s) - \frac{1}{s^2}F(s) = \frac{4}{s-1} \quad ,$$

So

$$F(s)(1 - \frac{1}{s^2}) = \frac{4}{s-1} \quad ,$$

Common denominator:

$$F(s)(\frac{s^2-1}{s^2}) = \frac{4}{s-1} \quad ,$$

giving

$$F(s) = \frac{4s^2}{(s^2-1)(s-1)} \quad .$$

We need to find

$$f(t) = \mathcal{L}^{-1}\{\frac{4s^2}{(s^2-1)(s-1)}\} \quad .$$

If we had Maple, we can just ask it, and be done. Since we don't, we have to go on. First factorize the denominator completely

$$\frac{4s^2}{(s^2-1)(s-1)} = \frac{4s^2}{(s-1)(s+1)(s-1)} = \frac{4s^2}{(s-1)^2(s+1)} \quad .$$

We need to do **partial fraction decomposition** with the template

$$\frac{4s^2}{(s-1)^2(s+1)} = \frac{A}{s+1} + \frac{B}{(s-1)} + \frac{C}{(s-1)^2} \quad .$$

Taking common denominator on the right side:

$$\frac{4s^2}{(s-1)^2(s+1)} = \frac{A(s-1)^2 + B(s-1)(s+1) + C(s+1)}{(s-1)^2(s+1)} \quad .$$

The denominators match automatically, so equting the numerators:

$$4s^2 = A(s-1)^2 + B(s-1)(s+1) + C(s+1) \quad .$$

When $s = 1$ we get $4 = C(2)$ so $C = 2$. When $s = -1$ we get $4 = A(-2)^2$ so $A = 1$. Expanding and comparing the coefficient of s^2 gives $4 = A + B$, so $B = 3$. Going back to the template, we have

$$F(s) = \frac{1}{s+1} + \frac{3}{s-1} + \frac{2}{(s-1)^2} \quad .$$

Now we are ready to apply $\mathcal{L}^{-1}$ in order to decipher $f(t)$, by using the table:

$$f(t) = \mathcal{L}^{-1}\{\frac{1}{s+1} + \frac{3}{s-1} + \frac{2}{(s-1)^2}\} = e^{-t} + 3e^t + 2te^t \quad .$$

11. (15 points) Solve the initial value problem

$$y'' + 2y' + 2y = \delta(t - \pi) \quad , \quad y(0) = 0 \quad , \quad y'(0) = 0 \quad .$$

Ans.:

$$y(t) = -e^{\pi - t} \sin t \, \mathcal{U}(t - \pi) \quad .$$

We first apply $\mathcal{L}$ to both sides

$$\mathcal{L}\{y'' + 2y' + 2y\} = \mathcal{L}\{\delta(t - \pi)\}$$

Getting

$$s^2 Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + 2Y(s) = e^{-\pi s} \quad .$$

Since $y(0) = 0, y'(0) = 0$, we get:

$$s^2 Y(s) + 2sY(s) + 2Y(s) = e^{-\pi s} \quad .$$

Facotorizing:

$$(s^2 + 2s + 2)Y(s) = e^{-\pi s} \quad .$$

Solving for $Y(s)$, we have:

$$Y(s) = \frac{e^{-\pi s}}{s^2 + 2s + 2} \quad .$$

Let's call

$$F(s) = \frac{1}{s^2 + 2s + 2} \quad .$$

Completing the square at the bottom"

$$F(s) = \frac{1}{s^2 + 2s + 2} = \frac{1}{(s + 1)^2 + 1} \quad .$$

from the table

$$f(t) = \mathcal{L}^{-1}\{\frac{1}{(s + 1)^2 + 1}\} = e^{-t} \sin t \quad .$$

Finally, also from the table ($a = \pi$)

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = (t - \pi)\mathcal{U}(t - \pi) = e^{-(t-\pi)} \sin(t - \pi)\mathcal{U}(t - \pi) \quad .$$

Using the trig identity $\sin(t - \pi) = -\sin t$ we get the answer above.

12. (15 points) Find the Fourier integral representation of

$$f(x) = \begin{cases} 0, & \text{if } x < -\pi; \\ 1, & \text{if } -\pi \le x < 0; \\ 2, & \text{if } 0 \le x \le \pi; \\ 0, & \text{if } x > \pi; \end{cases}$$

Ans.:

$$\frac{3}{\pi}\int_0^\infty \frac{\sin(\pi\alpha)}{\alpha}\cos\alpha x\, d\alpha + \frac{1}{\pi}\int_0^\infty \frac{1-\cos(\pi\alpha)}{\alpha}\sin\alpha x\, d\alpha \quad .$$

$f(x)$ is neither even nor odd, so we need both $A(\alpha)$ and $B(\alpha)$.

$$A(\alpha) = \int_{-\infty}^{\infty} f(x)\cos\alpha x\, dx = \int_{-\pi}^{0} (1)\cos\alpha x\, dx + \int_0^{\pi} (2)\cos\alpha x\, dx$$

$$= \frac{\sin\alpha x}{\alpha}\Big|_{-\pi}^{0} + 2\frac{\sin\alpha x}{\alpha}\Big|_0^{\pi}$$

$$= \frac{0-\sin(-\alpha\pi))}{\alpha} + 2\frac{\sin(\alpha\pi)) - 0}{\alpha} = \frac{\sin(\alpha\pi))}{\alpha} + 2\frac{\sin(\alpha\pi)) - 0}{\alpha} = 3\frac{\sin\alpha\pi}{\alpha} \quad .$$

Now we need to do $B(\alpha)$:

$$B(\alpha) = \int_{-\infty}^{\infty} f(x)\sin\alpha x\, dx = \int_{-\pi}^{0} (1)\sin\alpha x\, dx + \int_0^{\pi} (2)\sin\alpha x\, dx$$

$$= -\frac{\cos\alpha x}{\alpha}\Big|_{-\pi}^{0} - 2\frac{\cos\alpha x}{\alpha}\Big|_0^{\pi}$$

$$= -\frac{1-\cos(-\alpha\pi))}{\alpha} - 2\frac{\cos(\alpha\pi)) - 1}{\alpha} = \frac{-1+\cos(\alpha\pi))}{\alpha} + \frac{-2\cos(\alpha\pi)) + 2}{\alpha} = \frac{1-\cos\alpha\pi}{\alpha} \quad .$$

Since

$$f(x) = \frac{1}{\pi}\int_0^\infty A(\alpha)\cos\alpha x + \frac{1}{\pi}\int_0^\infty B(\alpha)\sin\alpha x \quad ,$$

we get the above answer.

13. (10 points) Approximate, with mesh-size $h = 1$, the solution of the boundary-value problem

$$u_{xx} + u_{yy} = 0 \quad , \quad 0 < x < 2 \quad , \quad 0 < y < 2 \quad ;$$

subject to the boundary conditions

$$u(0, y) = 1 \quad , \quad 0 < y < 2 \quad ; \quad u(2, y) = 4 \quad , \quad 0 < y < 2 \quad ;$$

$$u(x, 0) = 2 \quad , \quad 0 < x < 2 \quad ; \quad u(x, 2) = -2 \quad , \quad 0 < x < 2 \quad .$$

Ans.: The approximation of $u(1, 1)$ is: $\frac{5}{4}$.

We have, from the boundary conditions (since $h = 1$)

$$u_{0,1} = u(0, 1) = 1 \quad , \quad u_{1,2} = u(1, 2) = -2 \quad , \quad u_{2,1} = u(2, 1) = 4 \quad , \quad u_{1,0} = u(1, 0) = 2 \quad .$$

In this problem there is only one interior point, $(1, 1)$, and there is only one equation

$$u_{1,1} = \frac{u_{0,1} + u_{1,2} + u_{2,1} + u_{1,0}}{4} \quad ,$$

and one unknown $u_{1,1}$. Plugging the above values, we have

$$u_{1,1} = \frac{1 + (-2) + 4 + (2)}{4} = \frac{5}{4} \quad .$$

So $u(1, 1)$ is approximately $\frac{5}{4}$. (Note: this is a very bad approximation, since $h = 1$ is so big.)

14. (10 points) Find all the eigenvalues of the matrix

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \quad ,$$

and determine a basis for each eigenspace.

Ans.: Smallest eigenvalue: 0 Corresponding eigenvector: $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Largest eigenvalue: 4 Corresponding eigenvector: $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$$\det \begin{bmatrix} 2-\lambda & 1 \\ 4 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 - (1)(4) = (\lambda-2)^2 - 4 = \lambda^2 - 4\lambda + 4 - 4 = \lambda^2 - 4\lambda$$

So the **characteristic equation** is

$$\lambda^2 - 4\lambda = 0 \quad .$$

Solving it:

$$\lambda(\lambda - 4) = 0 \quad .$$

gives $\lambda = 0$ and $\lambda = 4$ as the two **eigenvalues**.

For each of these we need to find the corresponding eigenvectors.

When $\lambda = 0$ we have to find a vector $\begin{bmatrix} a \\ b \end{bmatrix}$ such

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad .$$

Doing the matrix-multiplication, we get two equations

$$2a + b = 0 \quad , \quad 4a + 2b = 0 \quad .$$

But the second is a multiple of the first, so we can discard it, and get that the general solution is $b = -2a$. Plugging this into the template $\begin{bmatrix} a \\ -2a \end{bmatrix} = a \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Taking $a = 1$ (we can take any **non-zero** value for a) gives the eigenvector for $\lambda = 0$.

When $\lambda = 4$ we have to find a vector $\begin{bmatrix} a \\ b \end{bmatrix}$ such

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 4 \begin{bmatrix} a \\ b \end{bmatrix} \quad .$$

Doing the matrix-multiplication, we get two equations

$$2a + b = 4a \quad , \quad 4a + 2b = 4b \quad .$$

That simplify to

$$-2a + b = 0 \quad , \quad 4a - 2b = 0 \quad .$$

But the second is a multiple of the first, so we can discard it, and get that the general solution is $b = 2a$. Plugging this into the template $\begin{bmatrix} a \\ 2a \end{bmatrix} = a \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Taking $a = 1$ (we can take any **non-zero** value for a) gives the eigenvector for $\lambda = 4$.