Dr. Z.'s Calc5 Lecture 7 Handout: Orthogonal Functions

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Important Definition 1

Two functions $f(x)$ and $g(x)$ defined on an interval $[a,b]$ are **orthogonal** if

$$\int_a^b f(x)g(x)\,dx = 0 \quad .$$

Important Definition 1'

Two functions $f(x)$ and $g(x)$ defined on an interval $[a,b]$ are **orthogonal** with respect to the **weight function** $w(x)$ if

$$\int_a^b f(x)g(x)\,w(x)dx = 0 \quad .$$

Comment: When the weight function $w(x)$ is the constant function 1 ($w(x)=1$), then Definition 1' becomes Definition 1.

Important Definition 2: A set of functions $\phi_1(x), \phi_2(x), \phi_3(x), \quad$ is an **orthogonal set** over $[a,b]$ if they are all orthogonal to each other. In other words

$$\int_a^b \phi_m(x)\phi_n(x)\,dx = 0 \quad whenver \quad m \neq n \quad .$$

Important Definition 2': A set of functions $\phi_1(x), \phi_2(x), \phi_3(x), \quad$ is an **orthogonal set** over $[a,b]$ with respect to the **weight** function $w(x)$ if the ϕ_i's are all orthogonal to each other, with respect to $w(x)$. In other words

$$\int_a^b \phi_m(x)\phi_n(x)\,w(x)dx = 0 \quad whenver \quad m \neq n \quad .$$

Comment: When the weight function $w(x)$ is the constant function 1 ($w(x)=1$), then Definition 2' becomes Definition 2.

Definition 3: The **inner-product** of two functions $(f(x), g(x))$ over $[a,b]$ is

$$(f,g) = \int_a^b f(x)g(x)\,dx \quad .$$

Definition 3': The **inner-product** of two functions $(f(x), g(x))$ over $[a,b]$ with respect to the weight function $w(x)$ is

$$(f,g)_w = \int_a^b f(x)g(x)\,w(x)dx \quad .$$

Comment: When the weight function $w(x)$ is the constant function 1 ($w(x) = 1$), then Definition 3' becomes Definition 3.

Definition 4: The **norm-squared** of a function $f(x)$ on an interval $[a, b]$ is

$$||f||^2 = (f, f) = \int_a^b f(x)^2 \, dx \quad .$$

Definition 4': The **norm-squared** of a function $f(x)$ on an interval $[a, b]$ with respect to the weight-function $w(x)$ is

$$||f||_w^2 = (f, f)_w = \int_a^b f(x)^2 \, w(x) dx \quad .$$

Important Theorem: **Any** (not too crazy) function $f(x)$ on $[a, b]$ can be written as a "linear combination" of an orthogonal set $\{\phi_1(x), \phi_2(x), \phi_3(x), \ldots\}$ (with respect to $w(x)$)

$$f(x) = c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x) + \ldots \quad ,$$

where the coefficients $c_1, c_2, c_3, \ldots$ standing in front of the ϕ_i's may be computed by

$$c_n = \frac{(f, \phi_n)}{(\phi_n, \phi_n)} \quad ,$$

Or more explicitly

$$c_n = \frac{\int_a^b f(x)\phi_n(x) \, w(x) dx}{\int_a^b \phi_n(x)^2 \, w(x) dx} \quad ,$$

Problem 7.1: Show that the given functions are orthogonal on the given interval.

$$f_1(x) = e^x x^3 \quad , \quad f_2(x) = e^{-x}(x^2 + 1) \quad , \quad [-1, 1] \quad .$$

Solution:

$$(f_1, f_2) = \int_{-1}^1 f_1(x) f_2(x) \, dx = \int_{-1}^1 (e^x x^3)(e^{-x}(x^2 + 1)) \, dx = \int_{-1}^1 e^{x-x} x^3 (x^2 + 1) \, dx$$

$$= \int_{-1}^1 x^3 (x^2 + 1) \, dx = \int_{-1}^1 (x^5 + x^3) \, dx =$$

$$\left(\frac{x^6}{6} + \frac{x^4}{4} \right) \Big|_{-1}^1 = \left(\frac{1^6}{6} - \frac{(-1)^6}{6} + \frac{1^4}{4} - \frac{(-1)^4}{4} \right) = 0 \quad .$$

Problem 7.2: Prove that the following set of functions is orthogonal on the indicated interval. Find the norm of each function in the set.

$$\{\cos x, \cos 3x, \cos 5x, \ldots, \cos(2n + 1)x, \ldots, \} \quad ; \quad [0, \pi/2] \quad .$$

Solution: Here a **typical** function is

$$\phi_n(x) = \cos(2n+1)x \quad .$$

Taking two **different** such functions $\phi_n(x)$, $\phi_m(x)$, we have

$$\phi_n(x) = \cos(2n+1)x \quad , \quad \phi_m(x) = \cos(2m+1)x \quad ,$$

(of course $m \neq n$). The inner product of these is:

$$(\phi_n, \phi_m) = \int_0^{\pi/2} (\cos(2n+1)x)(\cos(2m+1)x)\ dx \quad .$$

Recall the famous trig. identity

$$\cos A \cos B = \frac{\cos(A+B) + \cos(A-B)}{2} \quad .$$

So

$$(\phi_n, \phi_m) = \int_0^{\pi/2} \cos(2n+1)x \cos(2m+1)x = \frac{1}{2}\int_0^{\pi/2} (\cos(2n+1+2m+1)x + \cos(2n+1-2m-1)x)\, dx$$

$$= \frac{1}{2}\int_0^{\pi/2} (\cos(2n+2m+2)x + \cos(2n-2m)x)\, dx$$

$$= \frac{1}{2}\left(\frac{sin(2n+2m+2)x}{2(n+m+1)} + \frac{sin(2n-2m)x}{2(n-m)}\right)\Big|_0^{\pi/2} =$$

$$\frac{1}{2}\left(\frac{sin(2n+2m+2)(\pi/2)}{2n+2m+2} + \frac{sin(2n-2m)(\pi/2)}{2n-2m}\right) - \frac{1}{2}\left(\frac{sin(2n+2m+2)(0)}{2n+2m+2} + \frac{sin(2n-2m)(0)}{2n-2m}\right)$$

$$= \frac{1}{2}\left(\frac{sin(n+m+1)\pi}{2n+2m+2} + \frac{sin(n-m)\pi}{2n-2m}\right) - \frac{1}{2}\left(\frac{sin(2n+2m+2)(0)}{2n+2m+2} + \frac{sin(2n-2m)(0)}{2n-2m}\right) = 0-0 = 0 \quad .$$

Since $(n-m \neq 0)$.

The norm-squared of $\phi_n(x)$ is:

$$||\phi_n||^2 = (\phi_n, \phi_n) = \int_0^{\pi/2} \cos^2(2n+1)x\, dx \quad .$$

Recall another trig identity:

$$\cos^2 A = \frac{1+\cos 2A}{2} \quad .$$

So

$$||\phi_n||^2 = (\phi_n, \phi_n) = \int_0^{\pi/2} \cos^2(2n+1)x\, dx = \frac{1}{2}\int_0^{\pi/2} (1+\cos 2(2n+1)x)\, dx = \frac{1}{2}\left(x + \frac{\sin 2(2n+1)x}{4n+2}\right)\Big|_0^{\pi/2} =$$

$$\frac{1}{2}\left(\pi/2 + \frac{\sin 2(2n+1)(\pi/2)}{4n+2}\right) - \frac{1}{2}\left(0 + \frac{\sin 2(2n+1)(0)}{4n+2}\right) = \pi/4 \quad .$$

Finally the **norm** is the squared-root:

$$||\phi_n|| = \sqrt{\pi/4} = \frac{\sqrt{\pi}}{2} \quad .$$

Ans. to 7.2: The set is indeed orthogonal, and the norm of $\cos(2n+1)x$ is $\frac{\sqrt{\pi}}{2}$

Problem 7.3: Verify by direct integration that the functions are orthogonal with respect to the indicated weight function on the given interval

$$L_0(x) = 1 \quad , \quad L_1(x) = -x + 1 \quad , \quad ; w(x) = e^{-x} \quad ; [0, \infty) \quad .$$

Solution of 7.3:

$$(L_0, L_1) = \int_0^\infty 1(-x+1)e^{-x}\,dx = \int_0^\infty (-x+1)e^{-x}\,dx \quad .$$

By integration by parts (let's first do the indefinite integration)

$$\int (-x+1)e^{-x}\,dx = \int (-x+1)(-e^{-x})'\,dx = (-x+1)(-e^{-x}) - \int (-x+1)'(-e^{-x})$$

$$= (-x+1)(-e^{-x}) - \int e^{-x} = (-x+1)(-e^{-x}) + e^{-x} = xe^{-x} \quad .$$

So

$$\int_0^\infty (-x+1)e^{-x}\,dx = xe^{-x}\Big|_0^\infty = \infty \cdot e^{-\infty} - 0 \cdot e^{-0} = 0 - 0 = 0$$

(Note $\infty e^{-\infty}$ is shorthand for $\lim_{L\to\infty} Le^{-L}$ that is well known to be 0.)

Problem 7.4: (a) Show that the following set of functions, over the given interval and weight function is an orthogonal set.

$$\{\, f_1(x) = x \quad , \quad f_2(x) = 4x - 3 \quad , \quad f_3(x) = 10x^2 - 12x + 3 \,\}, \quad [0,1] \quad , \quad w(x) = x \quad .$$

(b) Using **orthogonality** (no credit for other methods!) find numbers c_1, c_2, c_3 such that

$$10x^2 = c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) \quad .$$

(c) Check directly the answer to part (b).

Solution of Part a:

$$(f_1, f_2)_w = \int_0^1 f_1(x) f_2(x) x\,dx = \int_0^1 x(4x-3)x\,dx = \int_0^1 (4x^3 - 3x^2)\,dx$$

$$= x^4 - x^3\Big|_0^1 = (1^4 - 1^3) - (0^4 - 0^3) = 0 \quad .$$

$$(f_1, f_3)_w = \int_0^1 f_1(x) f_3(x) x\, dx = \int_0^1 x(10x^2 - 12x + 3)x\, dx = \int_0^1 (10x^4 - 12x^3 + 3x^2)\, dx$$

$$= 2x^5 - 3x^4 + x^3\Big|_0^1 = (2(1)^5 - 3(1)^4 + (1)^3) - (2(0)^5 - 3(0)^4 + (0)^3) - = 2 - 3 + 1 - 0 = 0 \quad .$$

$$(f_2, f_3)_w = \int_0^1 f_2(x) f_3(x) x\, dx = \int_0^1 (4x-3)(10x^2 - 12x + 3)x\, dx = \int_0^1 (40x^4 - 78x^3 + 48x^2 - 9x)\, dx$$

$$= (8x^5 - \frac{39}{2}x^4 + 16x^3 - \frac{9}{2}x^2)\Big|_0^1 = (8\cdot(1)^5 - \frac{39}{2}\cdot(1)^4 + 16\cdot(1)^3 - \frac{9}{2}\cdot(1)^2) - 0 = 8 - \frac{39}{2} + 16 - \frac{9}{2} = 24 - \frac{48}{2} = 24 - 24 = 0 \quad .$$

Since each *pair* has inner-product 0, the system is indeed orthogonal with respect to the weight-function $w(x) = x$ over the interval $[0, 1]$.

Solution to Part b: Recall that if, for some numbers c_1, c_2, c_3 we have

$$F(x) = c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) \quad ,$$

then c_i can be gotten from

$$c_i = \frac{(F, f_i)_w}{(f_i, f_i)_w} = \frac{\int_a^b F(x) f_i(x) w(x)\, dx}{\int_a^b f_i^2(x) w(x)\, dx} \quad .$$

In this problem, $F(x) = 10x^2$, $f_1(x) = x, f_2(x) = 4x - 3, f_3(x) = 10x^2 - 12x + 3$. Also $w(x) = x$ and $[a, b] = [0, 1]$. So

$$c_1 = \frac{\int_0^1 (10x^2)(x) x\, dx}{\int_0^1 (x)^2 x\, dx} = \frac{\int_0^1 10x^4\, dx}{\int_0^1 x^3\, dx} = \frac{2}{1/4} = 8 \quad .$$

$$c_2 = \frac{\int_0^1 (10x^2)(4x-3) x\, dx}{\int_0^1 (4x-3)^2 x\, dx} = \frac{\int_0^1 (40x^4 - 30x^3)\, dx}{\int_0^1 (16x^3 - 24x^2 + 9x)\, dx} = \frac{1/2}{1/2} = 1 \quad .$$

$$c_3 = \frac{\int_0^1 (10x^2)(10x^2 - 12x + 3) x\, dx}{\int_0^1 (10x^2 - 12x + 3)^2 x\, dx} = \frac{\int_0^1 (100x^5 - 120x^4 + 30x^3)\, dx}{\int_0^1 (100x^5 - 240x^4 + 204x^3 - 72x^2 + 9x)\, dx} = \frac{1/6}{1/6} = 1 \quad .$$

Ans. to 7.4b): $\quad 10x^2 = 8 \cdot f_1(x) + 1 \cdot f_2(x) + 1 \cdot f_3(x) \quad .$

Solution to c):

$$8 \cdot f_0(x) + 1 \cdot f_1(x) + 1 \cdot f_2(x) = 8x + 4x - 3 + 3 - 12x + 10x^2 = 10x^2 \quad .$$

Yea! It worked out!