Dr. Z.'s Calc5 Lecture 5 Handout: The Dirac Delta Function

By Doron Zeilberger

Important Definition: The **Dirac Delta Function**, $\delta(t)$ is a "function" that is zero everywhere but ∞ at $t=0$. Its shift, $\delta(t-t_0)$ is zero everywhere and infinity at $t=t_0$.

Important Formula

$$\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0} \quad .$$

In paticular

$$\mathcal{L}\{\delta(t)\} = 1 \quad .$$

Problem 5.1: Solve the initial-value problem

$$y' + y = \delta(t-1) \quad , \quad y(0) = 2 \quad .$$

Solution: Applying $\mathcal{L}$:

$$\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{\delta(t-1)\} \quad ,$$

$\mathcal{L}\{y'\} = sY - y(0) = sY - 2$, $\mathcal{L}\{y\} = Y$, and $\mathcal{L}\{\delta(t-1)\} = e^{-s}$. So

$$sY - 2 + Y = e^{-s} \quad ,$$

$$(s+1)Y = 2 + e^{-s} \quad ,$$

$$Y = \frac{2}{s+1} + \frac{e^{-s}}{s+1} \quad .$$

$$y(t) = \mathcal{L}^{-1}\{\frac{2}{s+1}\} + \mathcal{L}^{-1}\{\frac{e^{-s}}{s+1}\} = 2e^{-t} + \mathcal{L}^{-1}\{\frac{e^{-s}}{s+1}\}$$

Recall that $\mathcal{L}^{-1}(e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a)$. Here $F(s) = \frac{1}{s+1}$ so $f(t) = e^{-t}$, and $a = 1$ so

$$\mathcal{L}^{-1}\{\frac{e^{-s}}{s+1}\} = e^{-(t-1)}\mathcal{U}(t-1) \quad .$$

So we have:

$$y(t) = 2e^{-t} + \mathcal{L}^{-1}\{\frac{e^{-s}}{s+1}\} = 2e^{-t} + e^{-(t-1)}\mathcal{U}(t-1) \quad .$$

Ans. to 5.1: $y(t) = 2e^{-t} + e^{-(t-1)}\mathcal{U}(t-1)$.

Problem 5.2: Solve the initial-value problem

$$y'' - 2y' = 1 + \delta(t-2) \quad , \quad y(0) = 0 \quad , \quad y'(0) = 0 \quad .$$

Solution: Applying $\mathcal{L}$:

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} = \mathcal{L}\{1\} + \mathcal{L}\{\delta(t-2)\} \quad ,$$

$\mathcal{L}\{y''\} = s^2Y - y(0)s - y'(0) = s^2Y$, $\mathcal{L}\{y'\} = sY - y(0) = sY$, $\mathcal{L}\{1\} = \frac{1}{s}$ and $\mathcal{L}\{\delta(t-2)\} = e^{-2s}$. So

$$s^2Y - 2sY = \frac{1}{s} + e^{-2s} \quad ,$$

$$(s^2 - 2s)Y = \frac{1}{s} + e^{-2s} \quad ,$$

$$s(s-2)Y = \frac{1}{s} + e^{-2s} \quad ,$$

$$Y = \frac{1}{s^2(s-2)} + \frac{e^{-2s}}{s(s-2)} \quad .$$

$$y(t) = \mathcal{L}^{-1}\{\frac{1}{s^2(s-2)}\} + \mathcal{L}^{-1}\{\frac{e^{-2s}}{s(s-2)}\} \quad .$$

Since $\mathcal{L}^{-1}\{\frac{1}{s-2}\} = e^{2t}$,

$$\mathcal{L}^{-1}\{\frac{1}{s(s-2)}\} = \int_0^t e^{2\tau}\, d\tau = \frac{1}{2}e^{2\tau}\Big|_0^t = \frac{1}{2}(e^{2t} - e^0) = \frac{1}{2}(e^{2t} - 1) \quad .$$

One more time

$$\mathcal{L}^{-1}\{\frac{1}{s^2(s-2)}\} = \frac{1}{2}\int_0^t (e^{2\tau} - 1)\, d\tau = \frac{1}{2}(\frac{e^{2\tau}}{2} - \tau)\Big|_0^t = \frac{1}{2}(\frac{e^{2t}}{2} - t - (\frac{e^0}{2} - 0)) = \frac{1}{4}(e^{2t} - 2t - 1)$$

We still need:

$$\mathcal{L}^{-1}\{\frac{e^{-2s}}{s(s-2)}\}$$

We already know (from above) that

$$\mathcal{L}^{-1}\{\frac{1}{s(s-2)}\} = \frac{1}{2}(e^{2t} - 1)$$

So

$$\mathcal{L}^{-1}\{\frac{e^{-2s}}{s(s-2)}\} = \frac{1}{2}(e^{2(t-2)} - 1)\mathcal{U}(t-2) \quad .$$

Combining, we get

$$y(t) = \frac{1}{4}(e^{2t} - 2t - 1) + \frac{1}{2}(e^{2(t-2)} - 1)\mathcal{U}(t-2) \quad .$$

Ans. to 5.2: $y(t) = \frac{1}{4}(e^{2t} - 2t - 1) + \frac{1}{2}(e^{2(t-2)} - 1)\mathcal{U}(t-2)$.