**Dr. Z.'s Calc5 Lecture 4 Handout: Operational Properties for the Laplace Transform**

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**Important Formula**

Once we know the Laplace transform $F(s)$ of some function $f(t)$ we can immediately figure out the Laplace transform of $t^n f(t)$ for *any* positive integer $n$

If $F(s) = \mathcal{L}\{f(t)\}$, and $n = 1, 2, 3, \ldots$, then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s) \quad .$$

**Problem 4.1**: Find $\mathcal{L}\{t \cos 4t\}$ **Solution**: Here $f(t) = \cos 4t$ and $n = 1$, so $F(s) = \frac{s}{s^2+16}$ and $n = 1$. The first derivative of $F(s)$ is, by the Quotient Rule

$$F'(s) = \frac{s'(s^2+16) - s(s^2+16)'}{(s^2+16)^2} = \frac{(1)(s^2+16) - s(2s)}{(s^2+16)^2} = \frac{s^2+16-2s^2}{(s^2+16)^2} = \frac{-s^2+16}{(s^2+16)^2} \quad .$$

So

$$\mathcal{L}\{t \cos 4t\} = (-1)^1 \frac{-s^2+16}{(s^2+16)^2} = \frac{s^2-16}{(s^2+16)^2} \quad .$$

**Ans. to 4.1**: $\mathcal{L}\{t \cos 4t\} = \frac{s^2-16}{(s^2+16)^2}$ .

**Problem 4.2**: Find $\mathcal{L}\{t^2 \sin 2t\}$ **Solution**: Here $f(t) = \sin 2t$ and $n = 2$, so $F(s) = \frac{2}{s^2+4} = 2(s^2+4)^{-1}$ and $n = 2$. The first derivative of $F(s)$ is (by the Chain Rule)

$$F'(s) = 2(-1)(s^2+4)^{-2} \cdot (2s) = \frac{-4s}{(s^2+4)^2} = (-4s)(s^2+4)^{-2} \quad .$$

By the Product Rule and the Chain Rule

$$F''(s) = (-4s)'(s^2+4)^{-2} + (-4s)((s^2+4)^{-2})' = (-4)(s^2+4)^{-2} + (-4s)(-2)(s^2+4)^{-3}(2s)$$

$$= (-4)(s^2+4)^{-2} + (16s^2)(s^2+4)^{-3} = \frac{-4}{(s^2+4)^2} + \frac{16s^2}{(s^2+4)^3} = \frac{-4(s^2+4)+16s^2}{(s^2+4)^3}$$

$$= \frac{-4s^2-16+16s^2}{(s^2+4)^3} = \frac{12s^2-16}{(s^2+4)^3} = \frac{4(3s^2-4)}{(s^2+4)^3} \quad .$$

So

$$\mathcal{L}\{t^2 \sin 2t\} = (-1)^2 \frac{4(3s^2-4)}{(s^2+4)^3} = \frac{4(3s^2-4)}{(s^2+4)^3} \quad .$$

**Ans. to 4.2**: $\mathcal{L}\{t^2 \sin 2t\} = \frac{4(3s^2-4)}{(s^2+4)^3}$.

**Important Definition**

The **Convolution** of two functions $f(t)$ and $g(t)$ written $f * g(t)$ is defined by the integral

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau \quad .$$

**Important Formulas**

If $f(t)$ and $g(t)$ are nice functions on $[0, \infty)$, then

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(s)G(s) \quad .$$

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t) \quad .$$

**Problem 4.3**: Evaluate

$$\mathcal{L}\{\int_0^t e^{2\tau}\cos(3(t-\tau))\,d\tau\} \quad .$$

**Solution.** The integral is the convolution $(f * g)(t)$ of $f(t) = e^{2t}$ and $g(t) = \cos(3t)$. From the tables we know

$$\mathcal{L}\{e^{2t}\} = \frac{1}{s-2} \quad , \quad \mathcal{L}\{\cos 3t\} = \frac{s}{s^2+9} \quad .$$

So

$$\mathcal{L}\{\int_0^t e^{2\tau}\cos(3(t-\tau))\,d\tau\} = \frac{1}{s-2}\cdot\frac{s}{s^2+9} = \frac{s}{(s-2)(s^2+9)} \quad .$$

**Ans. to 4.3**:$\mathcal{L}\{\int_0^t e^{2\tau}\cos(3(t-\tau))\,d\tau\} = \frac{s}{(s-2)(s^2+9)}$ .

**Problem 4.4**: Evaluate

$$\mathcal{L}^{-1}\{\frac{1}{(s+1)(s-1)}\} \quad ,$$

by using convolution (No credit for other methods [of course you can do it with partial fractions]).

**Solution**: Here

$$F(s) = \frac{1}{s+1} \quad , \quad G(s) = \frac{1}{s-1} \quad .$$

So

$$f(t) = e^{-t} \quad , \quad g(t) = e^{t} \quad .$$

$$f * g(t) = \int_0^t e^{-\tau}e^{t-\tau}\,d\tau = \int_0^t e^{-2\tau}e^{t}\,d\tau = e^t\int_0^t e^{-2\tau}\,d\tau$$

$$= e^t\frac{e^{-2\tau}}{-2}\Big|_0^t = e^t\left(\frac{e^{-2t}}{-2} - \frac{e^{-2\cdot 0}}{-2}\right) = \left(\frac{e^{-t}}{-2} - \frac{e^{t}}{-2}\right) = \frac{1}{2}(e^t - e^{-t}) \quad .$$

**Ans. to 4.4**: $\mathcal{L}^{-1}\{\frac{1}{(s+1)(s-1)}\} = \frac{1}{2}(e^t - e^{-t})$.

**Transform of an Integral**

An important special case is when $g(t) = 1$ and we get

$$\mathcal{L}\{\int_0^t f(\tau)\,d\tau\} = \frac{F(s)}{s} \quad ,$$

$$\mathcal{L}^{-1}\{\frac{F(s)}{s}\} = \int_0^t f(\tau)\,d\tau$$

**Problem 4.5**: Compute

$$\mathcal{L}\{\int_0^t \sin 5y\,dy\} \quad .$$

**Solution**: Here $f(t) = \sin 5t$, so $F(s) = \frac{5}{s^2+5}$ and

$$\mathcal{L}\{\int_0^t \sin 5y\,dy\} = \frac{\frac{5}{s^2+5}}{s} = \frac{5}{s(s^2+5)} \quad .$$

**Ans. to 4.5**:$\frac{5}{s(s^2+5)}$.

**Problem 4.6**: Compute

$$\mathcal{L}^{-1}\{\frac{1}{s(s^2+1)}\} \quad ,$$

without using partial fractions.

**Solution**: $F(s) = \frac{1}{s^2+1}$, so $f(t) = \sin t$, so

$$\mathcal{L}^{-1}\{\frac{1}{s(s^2+1)}\} = \int_0^t f(\tau)\,d\tau = \int_0^t \sin\tau\,d\tau = (-\cos\tau)\Big|_0^t = (-cos(t)) - (-cos0) = 1 - \cos t \quad .$$

**Ans. to 4.6**: $1 - \cos t$.

**Solving Integral Equations**

**Problem 4.7**: Use Laplace Transform to solve the following integral equation

$$f(t) + \int_0^t (t-\tau) f(\tau)\,d\tau = t \quad .$$

In convolution notation, this is

$$f + t * f = t$$

Applying $\mathcal{L}$ we get

$$\mathcal{L}\{f\} + \mathcal{L}\{t * f\} = \mathcal{L}\{t\} \quad .$$

So:

$$\mathcal{L}\{f\} + \mathcal{L}\{t\}\mathcal{L}\{f\} = \mathcal{L}\{t\} \quad .$$

Let, as usual $\mathcal{L}\{f\} = F$. Recall that $\mathcal{L}\{t\} = \frac{1}{s^2}$, so

$$F + F\frac{1}{s^2} = \frac{1}{s^2} \quad .$$

Doing the algebra to solve $F$:

$$F(1 + \frac{1}{s^2}) = \frac{1}{s^2}$$

$$F\left(\frac{1+s^2}{s^2}\right) = \frac{1}{s^2} \quad ,$$

so

$$F = \frac{1}{1+s^2} \quad .$$

So

$$f(t) = \mathcal{L}^{-1}\{\frac{1}{1+s^2}\} = \sin t \quad .$$

**Ans. to 4.7**: $f(t) = \sin t$.